SOLUBLE GROUPS SATISFYING THE MINIMAL CONDITION FOR NORMAL SUBGROUPS

by

Howard L. Silcock

A thesis submitted for the degree of

Doctor of Philosophy

in the

Australian National University

Canberra, March 1973.

STATEMENT

The work presented in this thesis is my own except where otherwise indicated.

Howard [illegible]

ACKNOWLEDGEMENTS

The research reported in this thesis was carried out under the supervision of Dr D. McDougall, and I should like to express my thanks to him for his patient help and encouragement, and for many useful comments on the draft versions of the thesis.

Several other members of staff and students at the Australian National University have also helped me to further my mathematical education in a variety of ways, ranging from personal conversations to lecture courses, and I am grateful to all these people for enabling me to benefit from their knowledge. I also thank Dr H. Heineken and Dr J.S. Wilson for providing a preprint of their recent joint paper.

I should like to express especial thanks to my parents who have given me much help and encouragement in the production of this thesis.

I am pleased to give credit to Mrs Barbara Geary for the excellent typing.

Finally I thank the Australian National University for the award of a research scholarship.

ABSTRACT

In this thesis we investigate the structural properties of certain classes of soluble groups satisfying the minimal condition for normal subgroups (which we abbreviate to Min-n), and we describe a technique for constructing a number of examples which illustrate some of the difficulties of obtaining a general theory for such groups. The impetus for these investigations was provided by work on metabelian groups satisfying Min-n published recently in a paper by McDougall [24] and in a subsequent paper by Hartley and McDougall [12].

Our first results are concerned with the class of metanilpotent groups satisfying Min-n . We show that much of the structure theory for metabelian groups satisfying Min-n developed in McDougall's paper can be carried over to the groups in this class. In particular we prove analogues of McDougall's results concerning Sylow subgroups and investigate conditions under which metanilpotent groups satisfying Min-n will split over their derived groups. We also describe a number of specific examples of metanilpotent groups satisfying Min-n to indicate the scope of these results.

For our next results we make use of the concept of a twisted wreath product to obtain a new description for certain types of metabelian groups satisfying Min-n . We show that by using this concept it is possible to describe the structure of a large class of metabelian groups satisfying Min-n in terms of the structure of certain subgroups which satisfy the stronger minimal condition for all subgroups (abbreviated to Min). In particular we obtain an alternative description in terms of twisted wreath products for some well-known examples due to Čarin [3] of metabelian groups which

satisfy Min-n but not Min .

In our final chapter we use a particular type of twisted wreath product introduced by Heineken and Wilson [13] for the construction of a number of examples of groups satisfying Min-n . These examples demonstrate, among other things, that the methods used for establishing our results on metanilpotent groups satisfying Min-n cannot be extended to give analogous results for the more general class of soluble groups of derived length three satisfying Min-n . In fact the examples even satisfy a rather stronger condition than Min-n , since in each one the normal subgroups form a well-ordered ascending chain. We then apply the same construction to exhibit examples of perfect locally soluble groups whose normal subgroups form well-ordered chains. The first example of a group of this type was constructed by McLain [26]; however, our method enables us to construct a large class of groups with these properties, containing in particular uncountably many pairwise non-isomorphic periodic Hopfian groups.

CONTENTS

CHAPTER ONE

INTRODUCTION

During the last fifty years much research has been done into the structure of algebraic systems satisfying various finiteness conditions, in particular the 'chain conditions' first studied in detail by Emmy Noether in the nineteen-twenties. Probably the most important results are to be found in the theory of rings, where Wedderburn's theory of semi-simple artinian rings provides a striking example of the far-reaching consequences that such conditions may have. However much progress has also been made in the study of the effects of chain conditions on the structure of groups, especially in the cases of soluble and nilpotent groups.

In particular, K.A. Hirsch, in a series of papers ([14]-[18]) in the years 1938-1954, initiated a study of soluble groups satisfying the maximal condition for subgroups. He showed that many results previously known to be true for finite soluble groups could be extended to the groups in this wider class. Subsequent work of Mal'cev [23] and others led to a number of remarkable results concerning groups satisfying the maximal condition for subgroups and other related finiteness conditions. These results also rely strongly on the groups involved being soluble, or at least satisfying one of the generalized versions of solubility and nilpotency introduced by Kuroš and Černikov [21].

The minimal condition for subgroups (which we henceforth abbreviate to Min) and other closely related finiteness conditions have also received a good deal of attention. In the case of soluble groups, a fundamental result is the theorem of Černikov [4] showing that every soluble group satisfying Min has an abelian normal subgroup of finite index. Since it is possible to give a complete characterization of abelian groups satisfying Min , this theorem gives a useful description which essentially reduces the study of soluble groups satisfying Min to the study of finite soluble groups.

The maximal and minimal conditions for normal subgroups (for which we use the usual abbreviations Max-n and Min-n respectively) are much weaker finiteness conditions, as they are satisfied, for example, by every simple group. In a well-known series of papers ([9]-[11]), Philip Hall proved a number of varied results concerning the properties of finitely generated soluble groups. These groups form a class strictly containing the class of soluble groups satisfying Max-n , as is shown in the first paper of the series. Since the appearance of these papers, several other papers have appeared dealing with properties of finitely generated soluble groups, and many of the questions raised by Hall have now been answered. However soluble groups satisfying Min-n have not been the subject of many papers, and there is relatively little known about these groups.

In this thesis our aim is to follow up some recent work on soluble groups satisfying Min-n which may be found in a paper by McDougall [24] and in a subsequent joint paper by Hartley and McDougall [12]. The results we have obtained form the subject matter of Chapters 3, 4 and 5, while Chapter 2 is devoted to a review of the basic concepts needed in later chapters, establishing notation and terminology and providing some background material for those concepts which are not completely standard.

Before describing the contents of Chapters 3, 4 and 5 we first outline some of the more important results which may be found in the literature concerning soluble groups satisfying Min-n .

One of the most important facts known about soluble groups satisfying Min-n is that these groups are locally finite. This was established by Baer [2] in 1964. An example constructed by McLain [26] in 1959 shows that locally soluble groups satisfying Min-n need not be soluble. However it has only very recently been shown by Heineken and Wilson [13] (as yet unpublished) that locally soluble groups satisfying Min-n can fail to be locally finite. Indeed Heineken and Wilson describe a method for constructing uncountably many pairwise non-isomorphic torsion-free locally soluble groups satisfying Min-n .

Another important, and easily proved, fact is that for nilpotent groups the condition Min-n is equivalent to Min . Moreover a result due to Baer [1] shows that the centre of a nilpotent group satisfying Min has finite index. The first examples of soluble groups satisfying Min-n but not Min were constructed by Čarin [3] in 1949. These groups are metabelian, each one being an extension of an elementary abelian p-group by a quasicyclic q-group, for distinct primes p and q , and they have no proper subgroups of finite index.

McDougall's paper [24] is chiefly concerned with properties of metabelian groups satisfying Min-n . McDougall shows, by means of a useful result due to Wilson [37], that every soluble group satisfying Min-n is a finite extension of a quasi-radicable group satisfying Min-n . (A group is *quasi-radicable* if it is generated by the n-th powers of its elements for each positive integer n .) He then proves that the Sylow p-subgroups (i.e. maximal p-subgroups) of a quasi-radicable metabelian group satisfying Min-n are abelian, for each prime p, and that such a group splits over its derived group. Moreover, he proves that every metabelian group satisfying Min-n is countable, and that every group which occurs as a subgroup of a metabelian group satisfying Min-n has a unique conjugacy class of Sylow π-subgroups (i.e. maximal π-subgroups) for every set π of primes.

In Chapter 3 we prove some results analogous to these for the class of metanilpotent groups satisfying Min-n. By one of the facts quoted above, every group in this class has a quasi-radicable subgroup of finite index which also satisfies Min-n. This subgroup is in fact nilpotent-by-abelian, since periodic quasi-radicable nilpotent groups are always abelian (see Corollary 3.26). We generalize the first of McDougall's results mentioned above by showing (Theorem 3.34) that the Sylow p-subgroups of a quasi-radicable nilpotent-by-abelian group G satisfying Min-n are nilpotent, for each prime p, and that their nilpotency class is bounded by the class of the derived group G'. In fact we show that each Sylow p-subgroup of G is a central product of a Sylow p-subgroup of G' and an abelian group. We have also indicated, in passing, how these properties may be established for the Sylow p-subgroups corresponding to a specific prime p when the condition Min-n is replaced by a weaker condition which we have called p-Min-n.

We next turn to proving an analogue of McDougall's theorem that every quasi-radicable metabelian group satisfying Min-n splits over its derived group. It is not possible to prove a direct analogue of this result, even in the case of centre-by-metabelian groups, as is shown by an example in McDougall's paper. Nevertheless we have proved that a partial analogue is true for all quasi-radicable nilpotent-by-abelian groups satisfying Min-n (Theorem 3.43). The difference is, roughly speaking, that one needs to allow for the possibility of forming certain central products which may destroy the splitting property. In particular, our result implies that the directly analogous result is valid whenever the centre of the group in question intersects the derived group trivially.

The other two results of McDougall mentioned above can both be extended without modification to the class of metanilpotent groups satisfying Min-n. The countability of the groups in this class follows from McDougall's result by a straightforward argument (Theorem 3.52), whereas the result concerning the conjugacy of Sylow π-subgroups follows immediately from Lemma 4.2 of Hartley and McDougall [12].

We close Chapter 3 by describing some specific examples of quasi-radicable nilpotent-by-abelian groups satisfying Min-n which show that this class of groups is really larger than the class of quasi-radicable metabelian groups satisfying Min-n. These examples show that there is no bound for the nilpotency class of the derived group of a group in the former class. They also establish incidentally that there is no bound for the derived length of a quasi-radicable soluble group satisfying Min-n, a fact established by a different method in McDougall's paper. These facts should be compared with the situation for soluble groups satisfying Min, where one can deduce immediately from the theorem of Černikov mentioned earlier that every quasi-radicable soluble group satisfying Min is abelian. We construct further examples of quasi-radicable soluble groups satisfying Min-n with arbitrarily large derived lengths in Chapter 5.

The results in Chapter 4 are ultimately based on work whose original aim was to analyse the structure of the groups of Čarin mentioned earlier, and to relate these groups to wreath products. Philip Hall showed in [9] that it is possible to obtain simple examples of soluble groups satisfying Max-n but not Max by forming wreath products of soluble groups satisfying Max (i.e. polycyclic groups). Unfortunately the analogous procedure of forming wreath products of soluble groups satisfying Min does not, in general, produce groups satisfying Min-n . Indeed it is not hard to show that the groups constructed in this way will satisfy Min-n only if they actually satisfy Min (see Lemma 2.42). Nevertheless it turned out from our analysis of Čarin's groups, and later of other more general examples of quasi-radicable metabelian groups satisfying Min-n , that these groups were in fact built up from groups satisfying Min in a different way, by a process of embedding certain metabelian groups satisfying Min into twisted wreath products. Moreover we were subsequently able to use these ideas to give a description for an arbitrary quasi-radicable metabelian group satisfying Min-n in terms of twisted wreath products and subgroups satisfying Min , although the description is rather more complicated in the general case.

The whole of Chapter 4 is taken up with obtaining a general description of this type for an arbitrary quasi-radicable metabelian group satisfying Min-n . The proof depends on an analysis of the systems of imprimitivity of certain modules, and we begin by discussing the concept of a system of imprimitivity and indicating its connection with twisted wreath products. For our analysis of the modules we rely heavily on results from the paper by Hartley and McDougall [12], which develops McDougall's work on metabelian groups satisfying Min-n a good deal further. This paper includes a classification of quasi-radicable metabelian groups satisfying Min-n in terms of certain irreducible modules for abelian groups, and it is these modules which we need to study in detail.

The indications of Chapter 4 are that the notion of a twisted wreath product is an important one for the study of soluble groups satisfying Min-n . This idea is reinforced in Chapter 5 where we make use of a special type of twisted wreath product for constructing further examples of groups satisfying Min-n .

Our principal aim in Chapter 5 is to construct examples of quasi-radicable soluble groups satisfying Min-n which lie outside the class of metanilpotent groups. The ideas behind the methods we employ are essentially those used by McLain [26] for the construction of the example mentioned earlier of a locally soluble group which satisfies Min-n but is not soluble. McLain's example is constructed using a certain tower of finite soluble groups, in each of which the only normal subgroups are the terms of the derived series and the only non-trivial central factor is the factor-group by the derived group. We make use of these finite soluble groups to construct the examples we require in the following way.

We first form the wreath product of any one of these groups, say A , with a quasicyclic group. Then we describe a method for embedding the resulting group into a group G such that G/G'' is isomorphic to one of Čarin's groups. This is done in such a way that the only normal subgroups of G below G'' are the normal closures of the terms of the derived series of A . It then follows easily that G is a quasi-radicable soluble group satisfying Min-n whose derived length exceeds that of A by two. Since McLain's tower of finite soluble groups contains soluble groups of arbitrarily large derived lengths, we thus obtain examples of quasi-radicable soluble groups satisfying Min-n of arbitrarily large derived lengths. Unlike the examples we construct in Chapter 3 these groups are not metanilpotent (except in the case $A = 1$); in fact the derived series of each of the groups coincides with the lower nilpotent series, so we have groups of arbitrarily large nilpotent lengths. Using this construction it is easy to produce examples of quasi-radicable soluble groups of derived length 3 with Sylow p-subgroups which are not even hypercentral. Thus the results of Chapter 3 concerning the nilpotency of Sylow p-subgroups cannot be extended to arbitrary quasi-radicable soluble groups of derived length 3 satisfying Min-n .

Rather than restricting ourselves to using the tower of finite soluble groups constructed by McLain, we have shown how to construct different towers of finite soluble groups with the properties described above. The existence of these also enables us to construct more groups with the properties of McLain's group: thus we establish as a by-product the (perhaps not very surprising) fact that McLain's group is not unique but one of a whole class of groups with similar properties. In fact we can easily prove in this way the existence of $2^{\aleph_0}$ pairwise non-isomorphic groups having properties similar to McLain's.

The presentation of the results in Chapter 5 has been substantially influenced by the ideas in the paper of Heineken and Wilson [13] referred to earlier, and we are much indebted to these authors for providing a preprint of their paper. In particular we have made use of the *treble product*, introduced in this paper, for the formulation of our results. Although this is a special case of the twisted wreath product, we have found that the notation employed enables us to express the ideas involved in the construction more clearly than was possible for our original formulation using twisted wreath products. Also we have been able to simplify the proof of one of our key lemmas (Lemma 5.11) by combining our ideas with those used by Heineken and Wilson for the proof of one of their lemmas.

NOTE. We use the symbol □ to indicate the end of a proof. This symbol is also placed immediately after the statement of any result for which no proof is included.

CHAPTER TWO

PRELIMINARIES

In this chapter we summarize the background material that we shall use in later chapters, often without specific references. The first section is devoted to notation and terminology; here we also record some fundamental facts of infinite group theory that we shall assume.

2.1 Notation and Terminology

Subgroups. Let G be a group. We use the following notation throughout:

$H \leq G$	H is a subgroup of G
$H \lhd G$	H is a normal subgroup of G
$\langle X \rangle$	the subgroup of G generated by the set X (with the usual convention concerning omission of braces)
$\langle H_\lambda : \lambda \in \Lambda \rangle$	the join of the subgroups H_λ $(\lambda \in \Lambda)$; i.e., the subgroup generated by $\bigcup_{\lambda \in \Lambda} H_\lambda$.

If π is a set of primes then a *π-subgroup* of G is a subgroup whose elements have finite orders divisible only by the primes in π . A *Sylow π-subgroup* of G is a maximal π-subgroup of G : the existence of such subgroups is guaranteed by Zorn's Lemma (which we assume throughout). When π consists of a single prime p then the terms *p-subgroup* and *Sylow p-subgroup* are used. As usual we use π' and p' for the sets of primes complementary to π and $\{p\}$ respectively.

We use 1 to denote the unit subgroup of any multiplicative group, as well as the identity element.

Conjugates and Commutators. If x and y are elements of a group G then we write as usual

$$x^y = y^{-1}xy \ ,$$

$$[x, y] = x^{-1}y^{-1}xy \ ,$$

and for elements $x_1, \ldots, x_n$ (where $n > 2$) we define recursively

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n] \ .$$

Similarly for subsets X, Y of G we write

$$X^Y = \langle x^y : x \in X, y \in Y \rangle \ ,$$

$$[X, Y] = \langle [x, y] : x \in X, y \in Y \rangle \ ,$$

and for subsets $X_1, \ldots, X_n$ (where $n > 2$) we define recursively

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n] \ .$$

The terms $\gamma_\alpha(G)$ of the *lower central series* of G are defined for each ordinal $\alpha > 0$ by

$\gamma_1(G) = G$,

$\gamma_\alpha(G) = [\gamma_{\alpha-1}(G), G]$ when α is a successor ordinal,

$\gamma_\alpha(G) = \bigcap_{\beta<\alpha} \gamma_\beta(G)$ when α is a limit ordinal.

The *centre* of a group G is denoted by $\zeta(G)$ and the terms $\zeta_\alpha(G)$ of the *upper central series* of G are defined for each ordinal α by

$\zeta_0(G) = 1$,

$\zeta_\alpha(G)/\zeta_{\alpha-1}(G) = \zeta(G/\zeta_{\alpha-1}(G))$ when α is a successor ordinal,

$\zeta_\alpha(G) = \bigcup_{\beta<\alpha} \zeta_\beta(G)$ when α is a limit ordinal.

For some ordinal α we have $\zeta_\alpha(G) = \zeta_{\alpha+1}(G)$ and we call this subgroup $\zeta_\alpha(G)$ the *hypercentre* of G.

The terms $G^{(\alpha)}$ of the *derived series* of G are defined for each ordinal α by

$$G^{(0)} = G ,$$

$$G^{(\alpha)} = [G^{(\alpha-1)}, G^{(\alpha-1)}] \quad \text{when } \alpha \text{ is a successor ordinal,}$$

$$G^{(\alpha)} = \bigcap_{\beta<\alpha} G^{(\beta)} \quad \text{when } \alpha \text{ is a limit ordinal.}$$

We also write $G' = G^{(1)} = \gamma_2(G)$ for the *derived group* of G.

If H is any subgroup of a group G then we denote the *centralizer* in G of H and the *normalizer* in G of H by

$$C_G(H) = \{g \in G : [g, h] = 1 \text{ for all } h \in H\} ,$$

$$N_G(H) = \{g \in G : h^g \in H \text{ for all } h \in H\} ,$$

respectively.

Generalized Solubility and Nilpotency. We shall not go into all the various generalizations of solubility and nilpotency which may be found for example in Kurosh [20] or Robinson [33]. Here we record only the definitions of the terms we shall actually use and some fundamental results which we assume. We take for granted the definitions and basic properties of soluble and nilpotent groups.

2.11 DEFINITION. A group is *hypercentral* (or a *ZA-group*) if it coincides with its hypercentre.

2.12 DEFINITION. A group is *locally soluble* (resp. *locally nilpotent*) if every finitely generated subgroup is soluble (resp. nilpotent).

The basic facts which we shall need concerning these concepts may be summarized in the following theorem.

2.13 THEOREM.

(i) (Mal'cev) *Every hypercentral group is locally nilpotent.*

(ii) (Hirsch-Plotkin) *In any group the product of the locally nilpotent normal subgroups is locally nilpotent.* □

Proofs of these facts may be found, for example, in Schenkman [35], pp. 204-206.

2.14 DEFINITION. Let G be a group. The *Hirsch-Plotkin radical* of G is the product of the locally nilpotent normal subgroups of G, and is denoted by $\rho(G)$.

The Hirsch-Plotkin radical is thus the unique maximal locally nilpotent normal subgroup of a group, by Theorem 2.13 *(ii)*.

Series. We shall not need the most general types of (transfinite) series for groups as defined in Kurosh [20] and Robinson [33]. Again we give the definitions only for the concepts we shall actually use.

2.15 DEFINITION. Let G be a group and ρ an ordinal number. An *ascending normal series* in G of *type* ρ is a set

$$S = \{G_\alpha : \alpha \le \rho\}$$

of normal subgroups of G satisfying

(i) $G_0 = 1$ and $G_\rho = G$,

(ii) $G_\alpha \le G_\beta$ whenever $\alpha \le \beta \le \rho$,

(iii) $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for all limit ordinals $\alpha \le \rho$.

If in addition S satisfies

(iv) $G_{\alpha+1}/G_\alpha$ is a minimal normal subgroup of G/G_α, for all $\alpha < \rho$,

then S is called an *ascending chief series* of G. More generally, if H and K are normal subgroups of G with $H < K$, then an *ascending G-chief series from* H *to* K is a set

$$S = \{G_\alpha : \alpha \leq \rho\}$$

of normal subgroups of G satisfying (ii), (iii) and (iv) such that

$$G_0 = H \text{ and } G_\rho = K .$$

An *ascending central series* is an ascending normal series

$$S = \{G_\alpha : \alpha \leq \rho\}$$

with the property that

$$G_{\alpha+1}/G_\alpha \leq \zeta(G/G_\alpha)$$

for all $\alpha < \rho$.

We should point out that our terminology follows that of Robinson [33] but is at variance with that of Kurosh [20] who uses the term *ascending normal series* in a different sense.

2.2 Groups satisfying minimal conditions

2.21 DEFINITION. Let G be a group admitting a set Ω of operators. We say that G satisfies the *minimal condition for* Ω*-subgroups* if every non-empty set of Ω-subgroups of G has a minimal element (under the partial ordering of set-theoretic inclusion).

This condition is equivalent to the requirement that every descending chain

$$G_1 \geq G_2 \geq G_3 \geq \ldots$$

of Ω-subgroups should have only a finite number of strict inclusions.

The cases of interest to us are

(i) Ω is empty: here every subgroup is an Ω-subgroup and we say that G satisfies the *minimal condition for subgroups.*

(ii) Ω is the group Inn G of inner automorphisms of G : here the Ω-subgroups are just the normal subgroups and we say that G satisfies the *minimal condition for normal subgroups*.

(iii) Ω is the group Aut G of all automorphisms of G : here the Ω-subgroups are just the characteristic subgroups and we say that G satisfies the *minimal condition for characteristic subgroups*.

(iv) G is a normal subgroup of some group A and Ω is the group of automorphisms induced on G by the inner automorphisms of A : here the Ω-subgroups are the normal subgroups of A contained in G and we say that G satisfies the *minimal condition for A-invariant subgroups*.

Henceforth we shall abbreviate these four conditions to

$$\text{Min, Min-}n\text{, Min-}c\text{, Min-}A$$

respectively.

Of these the condition Min-n is of course the most important to us.

The most elementary properties of groups satisfying Min-n are summarized in the lemmas below. We omit the proofs since the results are easy consequences of the definitions.

2.22 LEMMA. *Let G be a group and N a normal subgroup.*

(i) If G satisfies Min-n *then G/N satisfies* Min-n .

(ii) If G satisfies Min-n *then N satisfies* Min-c .

(iii) If N satisfies Min-G *and G/N satisfies* Min-n *then G satisfies* Min-n . □

2.23 LEMMA. *If the group G is a direct product*

$$G = \operatorname*{Dr}_{\lambda\in\Lambda} G_\lambda$$

then G satisfies Min-n *if and only if*

(i) *each direct factor* G_λ *satisfies* Min-n , *and*

(ii) *all but a finite number of the direct factors* G_λ *are trivial.* □

2.24 LEMMA. *Let* G *be a group satisfying* Min-n *and let* H *and* K *be normal subgroups with* $H < K$. *Then there is an ascending* G*-chief series from* H *to* K . □

Although Lemma 2.22 shows that the property Min-n is inherited by homomorphic images, it is not true that a subgroup of a group satisfying Min-n need inherit the property Min-n as we may show easily by the following example.

2.25 EXAMPLE. Let G be the alternating group (group of even finitary permutations) on the set of natural numbers. It is well-known that G is simple, and so certainly satisfies Min-n . However, the subgroup H defined by

$$H = \langle (1\ 2\ 3), (4\ 5\ 6), (7\ 8\ 9), \ldots \rangle$$

is clearly a direct product of an infinite number of cyclic groups of order 3 and therefore fails to satisfy Min-n by Lemma 2.23.

2.3 Quasi-radicability

A group G is said to be *radicable* (or *complete*) if it is always possible to extract roots within G : that is, if for every element $g \in G$ and every positive integer n there is an element $a \in G$ such that $a^n = g$. In the theory of abelian groups, where additive notation is usually used, this property is referred to as *divisibility* and divisible abelian groups are well-known.

For our purposes the following related property is more useful.

2.31 DEFINITION. Let G be a group and for each positive integer

n , write

$$G^n = \langle g^n : g \in G \rangle .$$

The group G is said to be *quasi-radicable* (or *semi-radicable* or *Černikov complete*) if $G = G^n$ for every $n > 0$.

It is clear that quasi-radicability is inherited by homomorphic images, and that every group generated by quasi-radicable subgroups is itself quasi-radicable.

The concept of quasi-radicability can be generalized in the following natural way.

2.32 DEFINITION. Let π be a set of primes. A group G is said to be *π-quasi-radicable* if $G = G^n$ for every π-number n . (If $\pi = \{p\}$ then we use the term *p-quasi-radicable*.)

A concept closely related to quasi-radicability is that of being *$\underline{\underline{F}}$-perfect*. Here $\underline{\underline{F}}$ denotes the class of finite groups and we use the term '$\underline{\underline{F}}$-perfect' as a special case of the following.

2.33 DEFINITION. Let $\underline{\underline{X}}$ be a class of groups. A group G is said to be *$\underline{\underline{X}}$-perfect* if no non-trivial homomorphic image of G belongs to $\underline{\underline{X}}$.

Thus a group is $\underline{\underline{F}}$-perfect if and only if it has no proper normal subgroups of finite index. Since every subgroup of finite index in a group contains a normal subgroup of finite index, an $\underline{\underline{F}}$-perfect group has also no proper subgroups of finite index.

The close relationship between these concepts for soluble groups is shown by the following lemma.

2.34 LEMMA. *Let G be a soluble group. The following are equivalent:*

(1) *G is $\underline{\underline{F}}$-perfect,*

(2) *G/G' is radicable,*

(3) *G is quasi-radicable.*

Proof. (1) ⇒ (2): Assume G is $\underline{\underline{F}}$-perfect. Every homomorphic image of G/G' is a homomorphic image of G; therefore G/G' is $\underline{\underline{F}}$-perfect. Since a non-trivial abelian group of finite exponent always has a non-trivial finite direct summand, it follows that G/G' is radicable.

(2) ⇒ (3): Assume G/G' is radicable and let n be a positive integer. Then

$$G/G' = (G/G')^n = G^nG'/G'$$

so

$$G = G^nG' .$$

Hence

$$G/G^n = G^nG'/G^n = (G/G^n)' .$$

As G/G^n is soluble, this implies that $G = G^n$. Therefore G is quasi-radicable.

(3) ⇒ (1): Assume G is quasi-radicable. If G had a proper normal subgroup N of finite index n, then we should have

$$G^n \leq N < G ,$$

contradicting the quasi-radicability of G. Therefore G is $\underline{\underline{F}}$-perfect. □

NOTE: The equivalence of (1) and (3) for hypercentral groups is proved in Kurosh [20], p. 234: however the argument given there applies equally well to arbitrary soluble groups. The equivalence of (2) and (3) is apparently a result of Gluškov [8].

The relevance of the ideas of this section to the study of groups satisfying Min-n is shown by the following result.

2.35 LEMMA. *Let* G *be a group satisfying* Min-n . *Then* G *has a unique normal* $\underline{\underline{F}}$*-perfect subgroup of finite index.*

Proof. As G satisfies Min-n there is a normal subgroup N of G minimal among the normal subgroups of finite index. Since the intersection of two subgroups of finite index is a subgroup of finite index, N is uniquely determined by this property.

Any subgroup M of N having finite index in N also has finite index in G and so contains a normal subgroup of G having finite index in G . But this implies that $M = N$. Hence N is $\underline{\underline{F}}$-perfect. □

2.4 Group-theoretical Constructions

We shall make use of a number of group-theoretical constructions in later chapters. Here we give a brief review of those that will occur, beginning with the familiar direct product, for which we need only establish our notation, and proceeding eventually to the twisted wreath product and the treble product, a construction only recently introduced in work of Heineken and Wilson (as yet unpublished). In the latter cases we give more detailed definitions, owing to the less familiar nature of the constructions.

In each case we may view the construction as a method for producing a new group from a given set of groups, such that the new group contains subgroups which are isomorphic to the given groups and are related to one another in some specified way. Often it is convenient to be able to identify the given groups with corresponding subgroups of the group constructed, and we shall indicate the cases where this may be done in a natural way.

It is also sometimes useful to describe the structure of a given group by showing that it may be constructed from certain subgroups using one of the constructions and making appropriate identifications. In this way one arrives, for example, at the notion of the 'internal' direct product; and there are similar 'internal' concepts corresponding to each of the other constructions.

Direct Products. If $\{A_\lambda : \lambda \in \Lambda\}$ is a family of (not necessarily distinct) groups, we denote the *direct product* of the groups in the family by

$$\operatorname*{Dr}_{\lambda\in\Lambda} A_\lambda .$$

If the groups all coincide with some group A then we have the *direct power*, for which we use the notation

$$A^{(\Lambda)} .$$

For each $\lambda \in \Lambda$ there are two canonical homomorphisms associated with a direct product $D = \operatorname*{Dr}_{\lambda\in\Lambda} A_\lambda$, the *projection* map $\pi_\lambda : D \to A_\lambda$ and the *injection* map $\kappa_\lambda : A_\lambda \to D$. Using these homomorphisms we may associate with each subgroup B of D two (not necessarily distinct) subgroups of A_λ, namely the image $B\pi_\lambda$ of B under π_λ and the inverse image $B\kappa_\lambda^{-1}$ of B under κ_λ. In the case of the internal direct product κ_λ is simply the inclusion map $A_\lambda \to D$, and $B\kappa_\lambda^{-1} = B \cap A_\lambda$. The subgroup $B\pi_\lambda$ is referred to as the *projection* of B in the factor A_λ. We have $B\kappa_\lambda^{-1} \leq B\pi_\lambda$ for each $\lambda \in \Lambda$; and B is a direct product of its projections in the factors if and only if these subgroups coincide for every λ.

Central Products. We shall not introduce any special notation for the *central product* (or *direct product with amalgamation*). We use only the internal version of this concept: thus we say that a group G is a central product of subgroups A and B if

(i) $G = \langle A, B \rangle$, and

(ii) $[A, B] = 1$.

Split Extensions. If A and B are groups and $\theta : B \to \operatorname{Aut} A$ is a homomorphism then we write

$$A \underset{\theta}{]} B$$

for the *split extension* (or *semi-direct product*) of A and B , formed according to the homomorphism θ . We shall sometimes omit the θ when it may be understood from the context.

If G is a group, we shall also use

$$G = A \,]\, B$$

to mean that A is a normal subgroup of G complemented by B (this is the 'internal' version of the split extension).

Wreath Products. By the *wreath product* of two groups A and B we shall mean the restricted standard wreath product as defined for example in [30]. We denote this by

$$A \text{ wr } B .$$

The base group of A wr B is a direct power $A^{(B)}$ and for each $b \in B$ we may define the coordinate subgroup A_b as in [30] by

$$A_b = \{f \in A^{(B)} : f(b') = 1 \text{ for all } b' \neq b\} .$$

We shall often identify A with the coordinate subgroup A_1 corresponding to the identity element of B . When this is done the coordinate subgroups are precisely the conjugates of A in B , and we may express the base group as a direct product

$$\underset{b \in B}{\mathrm{Dr}}\ A^b .$$

We shall make use of the following well-known result concerning automorphisms of wreath products (see [29], p. 476), in which we assume the above identification is made.

2.41 LEMMA. *If* α *and* β *are automorphisms of groups* A *and* B *respectively then there is an automorphism of* $A \text{ wr } B$ *extending both* α *and* β . $\square$

Although the wreath product is in general a useful method for constructing soluble groups, we now show that it is of little use for constructing interesting examples of soluble groups satisfying Min-n . The following result is possibly well-known, but does not seem to appear in the literature.

2.42 LEMMA. *Let* A *and* B *be groups satisfying* Min-n *and assume that* $A \neq A'$ *and that* B *is locally finite. The group* $G = A \text{ wr } B$ *satisfies* Min-n *if and only if* B *is finite.*

Proof. The sufficiency of the condition is easy to see, for if B is finite the base group of G is a direct product of a finite number of groups satisfying Min-n and so itself satisfies Min-n . But G is a finite extension of its base group so it will also satisfy Min-n .

To prove the necessity, we suppose B infinite and construct a descending chain of normal subgroups in G . Since $(A/A') \text{ wr } B$ is a homomorphic image of $A \text{ wr } B$, and hence will satisfy Min-n if G does, we may assume that A is abelian.

For any finite subgroup F of B , write

$$A^{(B:F)} = \left\{f \in A^{(B)} : f(b_1) = f(b_2) \text{ if } b_1b_2^{-1} \in F\right\} ;$$

thus $A^{(B:F)}$ consists of the functions in $A^{(B)}$ which are constant on the right cosets of F in B . This subgroup is normal in G : to show this we need only verify that it is normalized by B , in view of our assumption that A is abelian. But if $f \in A^{(B:F)}$ and $b, b_1, b_2 \in B$ with $b_1 b_2^{-1} \in F$, then

$$f^b(b_1) = f\left(b_1 b^{-1}\right)$$

$$= f\left(b_2 b^{-1}\right) \quad \text{since} \quad \left(b_1 b^{-1}\right)\left(b_2 b^{-1}\right)^{-1} = b_1 b_2^{-1} \in F$$

$$= f^b\left(b_2\right) .$$

Thus $f^b \in A^{(B:F)}$ and so $A^{(B:F)}$ is normal in G.

Now we show that the mapping which associates with each finite subgroup F of B the subgroup $A^{(B:F)}$ defined above is order-reversing and maps distinct finite subgroups of B to distinct normal subgroups of G.

Let F_1 and F_2 be finite subgroups of B. If $F_1 \leq F_2$ and $f \in A^{(B:F_2)}$ then, since $b_1 b_2^{-1} \in F_1$ implies $b_1 b_2^{-1} \in F_2$, we have also $f \in A^{(B:F_1)}$. Thus the mapping is order-reversing. Now suppose only that $F_1 \neq F_2$: we may assume that there is an element $b_1 \in F_1$ with $b_1 \notin F_2$. Let $1 \neq a \in A$ and define an element $f \in A^{(B)}$ by

$$f(b) = a \quad \text{if } b \in F_2 ,$$

$$f(b) = 1 \quad \text{otherwise.}$$

Clearly $f \in A^{(B:F_2)}$; but $f \notin A^{(B:F_1)}$ since

$$f\left(b_1\right) \neq f(1)$$

although $b_1 \in F_1$. Therefore $A^{(B:F_1)} \neq A^{(B:F_2)}$. Hence the mapping $F \longmapsto A^{(B:F)}$ has the properties claimed.

Since B is an infinite locally finite group, it has an infinite strictly ascending chain of finite subgroups; and the above shows that there is an infinite strictly descending chain of normal subgroups of G. Hence G does not satisfy Min-n.

This establishes the sufficiency and completes the proof. □

Twisted Wreath Products. The *twisted wreath product* is a generalization of the wreath product due to B.H. Neumann [28] and will be used frequently in subsequent chapters.

Let A and C be groups, let $B \leq C$, and suppose that $\theta : B \to \operatorname{Aut} A$ is a homomorphism. The twisted wreath product is constructed from this data as follows.

Choose a right transversal T to B in C and form the direct power $A^{(T)}$. Now define a homomorphism

$$\theta^* : C \to \operatorname{Aut} A^{(T)}$$

as follows. For any elements $f \in A^{(T)}$ and $c \in C$ denote by f^c the element of $A^{(T)}$ defined by

$$f^c(t) = f(t')^{(b'\theta)^{-1}}$$

where the elements $t' \in T$ and $b' \in B$ are those uniquely determined by the equation

$$tc^{-1} = b't' .$$

We define $c\theta^*$ to be the mapping under which f corresponds to f^c. It is routine to verify that $c\theta^*$ is an automorphism of $A^{(T)}$ and that θ^* is a homomorphism from C into $\operatorname{Aut} A^{(T)}$.

The twisted wreath product determined by A, B, C and θ is by definition the split extension

$$A^{(T)} \rbrack_{\theta^*} C$$

and we denote it by

$$A \operatorname{wr}_B C .$$

It is shown in [28] that this definition is independent of the choice of transversal T. It is obviously not, however, independent of the choice of homomorphism θ from B into $\operatorname{Aut} A$ and a more accurate notation would indicate the dependence on θ. Nevertheless in the cases we consider the homomorphism intended will be made clear by the context and it is more convenient simply to specify A, B and C. (*Cf.* the notation used by Huppert [19], p. 99.)

Following the terminology used for wreath products, we refer to $A^{(T)}$ as the *base group* and define *coordinate subgroups* A_t for each $t \in T$ by

$$A_t = \{f \in A^{(T)} : f(t') = 1 \text{ if } t' \neq t\} .$$

Since the choice of transversal does not affect the definition we shall usually assume that T contains the identity element of C. In this case it is natural to identify A with the coordinate subgroup A_1. For each $t \in T$ the coordinate subgroup A_t is then simply the conjugate A^t and we may express the base group as a direct product

$$\operatorname*{Dr}_{t \in T} A^t .$$

Treble Products. In Chapter 5 we shall make use of the *treble product*, introduced in a recent (and as yet unpublished) paper of Heineken and Wilson [13]. This is a special case of the twisted wreath product, as the authors point out; however we follow the authors in formulating the definition in a somewhat different manner.

Let A, B, C be three groups and let

$$\sigma : B \to \mathrm{Aut}\, A \ ,$$

$$\tau : C \to \mathrm{Aut}\, B$$

be homomorphisms. (We write b^{σ}, c^{τ} for the images of elements $b \in B$, $c \in C$ under the homomorphisms σ, τ respectively.) Let

$$W = A \text{ wr } C$$

and identify A with the 1-component of the base group of W , so that W is generated by the groups A and C . Now form the free product

$$F = W * B$$

of W and B ; this contains isomorphic copies of the factors W and B , and hence also of A and C by our above identification, and we again identify these copies with the original groups. Let N be the normal closure in F of the subgroup generated by all elements having either of the forms

(i) $a^{b^{\sigma}} b^{-1} a^{-1} b$, or

(ii) $b^{c^{\tau}} c^{-1} b^{-1} c$

where $a \in A$, $b \in B$, $c \in C$. The treble product of A, B and C , formed according to the homomorphisms σ, τ , is by definition the factor-group F/N , and is denoted by

$$\mathrm{Tr}(A, B, C; \sigma, \tau) \ ,$$

or simply by $\mathrm{Tr}(A, B, C)$ when the homomorphisms σ, τ are clearly specified by the context.

Since $W \cap N = B \cap N = 1$, the subgroups W and B of F are mapped isomorphically onto their images in $T = F/N$ by the natural projection $F \to F/N$. Hence T also contains an isomorphic copy of each of the groups W, A, B, C ; and once again we identify these copies with the originals. Thus T becomes a group generated by

A, B and C with the property that

$$\langle A, C\rangle = A \operatorname{wr} C .$$

From the construction it is evident that the defining relations of T are:

(i) those of the factors A, B, C ,

(ii) all relations of the form

$$\left[a_1, a_2^c\right] = 1 \quad \{a_1, a_2 \in A \text{ and } 1 \neq c \in C\}$$

(these express the property that $\langle A, C\rangle = A \operatorname{wr} C$),

(iii) all relations of the form

$$a^{b^\sigma} = b^{-1}ab \quad (a \in A,\ b \in B) ,$$

$$b^{c^\tau} = c^{-1}bc \quad (b \in B,\ c \in C) ,$$

arising from factoring out the normal subgroup N .

The third group of relations show that the subgroups $\langle A, B\rangle$ and $\langle B, C\rangle$ are isomorphic respectively to the split extensions $A \,]_\sigma\, B$, $B \,]_\tau\, C$.

Writing $H = BC$ we see that T is isomorphic to the twisted wreath product $A \operatorname{wr}_B H$, with B acting on A according to the homomorphism σ .

Heineken and Wilson have shown that it is possible to iterate the treble product construction to produce a *treble product tower*. We shall also use an iterative construction based on the treble product, though ours differs from Heineken and Wilson's. (We describe the treble product tower in section 5.4 for comparison.) For our construction we use the following analogue of Lemma 2.41.

2.43 LEMMA. *Let the group* G *be a treble product*

$$G = \operatorname{Tr}(A, B, C; \sigma, \tau)$$

and let α, β, γ *be automorphisms of* A, B, C *respectively. If these automorphisms satisfy the conditions*

(1) $b^{\beta\sigma} = \alpha^{-1}b^{\sigma}\alpha$ *for all* $b \in B$,

(2) $c^{\gamma\tau} = \beta^{-1}c^{\tau}\beta$ *for all* $c \in C$,

then there is an automorphism ϕ *of* G *which extends the automorphisms* α, β, γ .

Conversely, if ϕ *is an automorphism of* G *leaving each of the subgroups* A, B, C *invariant and if we denote by* α, β, γ *the restrictions of* ϕ *to* A, B, C *respectively, then* α, β, γ *satisfy* (1) *and* (2).

Proof. By Lemma 2.41, the automorphisms α and γ can be simultaneously extended to an automorphism, ψ say, of $W = \langle A, C \rangle = A \text{ wr } C$. Moreover by the universal property of free products there is a unique automorphism, χ say, of the free product

$$F = W * B$$

extending the automorphisms ψ and β . Now G is the factor-group F/N , where N is defined as at the beginning of this section; we show next that N is invariant under χ .

Let $a \in A$, $b \in B$, $c \in C$. Then

$$\begin{aligned}\left(a^{b^{\sigma}}b^{-1}a^{-1}b\right)^{\chi} &= \left(a^{b^{\sigma}}\right)^{\chi}(b^{\chi})^{-1}(a^{\chi})^{-1}b^{\chi} \\ &= \left(a^{b^{\sigma}}\right)^{\alpha}(b^{\beta})^{-1}(a^{\alpha})^{-1}b^{\beta} \\ &= (a^{\alpha})^{b^{\beta\sigma}}(b^{\beta})^{-1}(a^{\alpha})^{-1}b^{\beta} ,\end{aligned}$$

using condition (1). Thus the image under χ of a generator of type (i) for N is again of type (i) , with a^{α} and b^{β} in place of a and b respectively. Similarly using condition (2) we have

$$\left(b^{c^{\tau}}c^{-1}b^{-1}c\right)^{\chi} = (b^{\beta})^{c^{\gamma\tau}}(c^{\gamma})^{-1}(b^{\beta})^{-1}c^{\gamma}$$

and it follows that N is invariant under χ, as claimed.

Therefore χ induces an automorphism ϕ on $G = F/N$. Since the subgroups A, B, C are identified with their images under the projection $F \to F/N$ in the definition of the treble product, it follows that ϕ is an automorphism of G extending the given automorphisms α, β, γ as required.

To prove the converse, suppose ϕ is an automorphism of G leaving each of the factors A, B, C invariant, and let α, β, γ be its restrictions to A, B, C respectively. Let $a \in A$ and $b \in B$. Then

$$\begin{aligned} a^{b^{\beta\sigma}} &= \left(b^{\beta}\right)^{-1} a b^{\beta} \\ &= \left(b^{\phi}\right)^{-1} a b^{\phi} \\ &= \left(b^{-1} a^{\phi^{-1}} b\right)^{\phi} \\ &= a^{\phi^{-1} b^{\sigma} \phi} \\ &= a^{\alpha^{-1} b^{\sigma} \alpha} . \end{aligned}$$

Hence $b^{\beta\sigma} = \alpha^{-1} b^{\sigma} \alpha$, for all $b \in B$, and condition (1) is satisfied. The proof that (2) is satisfied is exactly similar. □

2.44 COROLLARY. *Let G be a split extension*

$$G = A \mathop{]}_{\sigma} B .$$

If α and β are automorphisms of A and B respectively such that

(1) $b^{\beta\sigma} = \alpha^{-1} b^{\sigma} \alpha$, *for all* $b \in B$,

then there is an automorphism ϕ of G extending both α and β.

Conversely, if ϕ is an automorphism of G leaving both A and B invariant, then the restrictions α and β of ϕ to A

and B *respectively satisfy* (1).

Proof. Take $C = 1$ in Lemma 2.43. $\square$

CHAPTER THREE

METANILPOTENT GROUPS SATISFYING MIN-$\mathcal{N}$

In this chapter we show that some results of McDougall [24] concerning metabelian groups satisfying Min-n have natural extensions to the class of metanilpotent groups satisfying Min-n. These are discussed in the Introduction; here we merely summarize the main points to indicate the organization of the sections in this chapter.

The first of McDougall's results that we generalize concerns Sylow p-subgroups. McDougall shows that the Sylow p-subgroups of every quasi-radicable metabelian group satisfying Min-n are abelian, for each prime p. The analogous result for metanilpotent groups is Theorem 3.34. The proof of this theorem depends on two results of a more general nature, which we establish in sections 3.1 and 3.2. The first of these concerns the properties of the Hirsch-Plotkin radical of a soluble group satisfying Min-n, and the second is a generalization of a theorem of Černikov on quasi-radicable hyper-central groups. We combine these results in section 3.3 to yield a proof of Theorem 3.34, as well as the slightly more general Theorem 3.33.

In section 3.4 we prove a theorem concerned with the splitting properties of metanilpotent groups satisfying Min-n. This is a partial analogue of a result of McDougall which shows that every quasi-radicable metabelian group satisfying Min-n splits over its derived group.

Section 3.5 consists of a proof that every metanilpotent group satisfying Min-n is countable: this is a straightforward deduction from the analogous result for metabelian groups proved by McDougall.

Finally in section 3.6 we consider some specific examples of quasi-radicable metanilpotent groups satisfying Min-n. These groups are necessarily nilpotent-by-abelian; however we shall see that there is no bound either to their derived lengths or to the nilpotency class of their derived groups.

3.1 The Hirsch-Plotkin Radical

The first step in analysing the structure of the Sylow p-subgroups of a metanilpotent group satisfying Min-n is to show that every Sylow p-subgroup is hypercentral. To do this we first show that the Hirsch-Plotkin radical of such a group is hypercentral. In fact this latter result is true for every soluble group satisfying Min-n, as we shall prove below.

We begin by proving a result which does not involve the condition Min-n, but is an analogue for locally finite groups of a well-known result in finite group theory.

Following the usual practice in finite group theory, we write $O_{p'}(G)$ to denote the largest normal p'-subgroup of a group G and $O_{p'p}(G)$ to denote the inverse image in G of the largest normal p-subgroup of $G/O_{p'}(G)$. It is well-known that if G is a finite group then $O_{p'p}(G)$ centralizes every p-chief factor of G. We shall prove that this is also true when G is any periodic locally soluble group. (By a well-known result of McLain [25], every chief factor of a locally soluble group is abelian.) This has also been proved by Gardiner, Hartley and Tomkinson [7], using an unpublished result due to Kegel. However we give an alternative proof below.

3.11 THEOREM. *Let* G *be a periodic locally soluble group and* p *a prime. Then* $O_{p'p}(G)$ *centralizes every* p*-chief factor of* G .

Proof. Let $N = O_{p'p}(G)$. To obtain a contradiction, we suppose that some p-chief factor H/K is not centralized by N . We may further suppose that $K = 1$, so that H is a minimal normal p-subgroup of G .

Let $a \in H$ and $x \in N$ be elements such that

$$b = [a, x] \neq 1 .$$

Then $b \in H$ and, since H is a minimal normal subgroup, we have

$$H = \langle a\rangle^G = \langle b\rangle^G .$$

As G is locally finite, there is a finite subgroup F of G such that

$$\langle a\rangle^F = \langle b\rangle^F$$

and we may suppose F is chosen to contain the elements a, b and x . Let us write

$$H_0 = \langle a\rangle^F = \langle b\rangle^F .$$

Now every finite subgroup of N is p-nilpotent, so we have

$$\langle x\rangle^F \leq O_{p'p}(F) .$$

Hence

$$\begin{aligned} H_0 = \langle b\rangle^F &\leq \left[\langle a\rangle^F, \langle x\rangle^F\right] \\ &\leq \left[H_0, O_{p'p}(F)\right] . \end{aligned}$$

However, H_0 is normal in F and consequently there is a normal subgroup K_0 of F such that H_0/K_0 is a chief factor of F . As H_0 is a p-group, this is a p-chief factor of F , and is therefore centralized by $O_{p'p}(F)$, since we know the theorem is valid for finite groups. We deduce that

$$H_0 \le [H_0, O_{p'p}(F)] \le K_0 < H_0 ,$$

and this is plainly a contradiction, so the theorem is proved. $\square$

3.12 COROLLARY. *Let G be a periodic locally soluble group. Then the Hirsch-Plotkin radical $\rho(G)$ centralizes every chief factor of G.*

Proof. Since $\rho(G)$ is locally nilpotent, it has a unique Sylow p'-subgroup, for every prime p. Hence we have

$$\rho(G) \le O_{p'p}(G)$$

for every prime p. $\square$

We now investigate what more can be said about the subgroups $O_{p'p}(G)$ and $\rho(G)$ of a periodic locally soluble group G when we assume in addition that G satisfies Min-n. In fact for what we shall prove about $O_{p'p}(G)$ it turns out to be sufficient to assume only the weaker condition p-Min-n which we define below. This condition does not seem to have been considered before, although Polovickiĭ [32] has investigated an analogous generalization of the condition Min.

3.13 DEFINITION. Let p be a prime. A group G is said to satisfy the *p-minimal condition for normal subgroups* (henceforth abbreviated to p-Min-n) if for every descending chain

$$G_0 > G_1 > G_2 > \dots$$

of normal subgroups of G all but a finite number of the factors G_{i-1}/G_i are p-torsion-free.

It is easy to verify that this condition is inherited by homomorphic images, and that a periodic group satisfies Min-n if and only if it satisfies p-Min-n for every prime p.

3.14 THEOREM. *Let G be a periodic locally soluble group.*

(i) If G satisfies p-Min-n for some prime p, then $O_{p'p}(G)/O_{p'}(G)$ is hypercentral.

(ii) If G satisfies Min-n, then $\rho(G)$ is hypercentral.

Proof. *(i)* Suppose G satisfies p-Min-n. Since $O_{p'p}(G)/O_{p'}(G)$ is a p-group, the normal subgroups of G between $O_{p'}(G)$ and $O_{p'p}(G)$ are well-ordered by inclusion. Hence there is an ascending G-chief series

$$O_{p'}(G) = H_0 < H_1 < H_2 < \ldots < H_\gamma = O_{p'p}(G)$$

from $O_{p'}(G)$ to $O_{p'p}(G)$. By Theorem 3.11 each of the p-chief factors $H_{\alpha+1}/H_\alpha$ is central in $O_{p'p}(G)$. Consequently $O_{p'p}(G)/O_{p'}(G)$ is hypercentral.

(ii) Suppose G satisfies Min-n. Then there is an ascending G-chief series

$$1 = H_0 < H_1 < H_2 < \ldots < H_\gamma = \rho(G)$$

from 1 to $\rho(G)$. By Corollary 3.12 each of the chief factors $H_{\alpha+1}/H_\alpha$ is central in $\rho(G)$. Consequently $\rho(G)$ is hypercentral. □

We now deduce from these results that the Sylow p-subgroups of a metanilpotent group satisfying Min-n are hypercentral, for each prime p. In the case of a metanilpotent group satisfying p-Min-n we can deduce in the same way that the Sylow p-subgroups (for this specific prime p) are hypercentral, provided we assume that the group is locally finite. This assumption is unnecessary in the former case, because we can invoke the following theorem of Baer.

3.15 THEOREM (Baer [2]). *Every soluble group satisfying Min-n is locally finite.* □

3.16 THEOREM. *Let* G *be a group having a normal subgroup* N *such that both* N *and* G/N *are locally nilpotent.*

(i) If G *is locally finite and satisfies* p-Min-n *for some prime* p *, then the Sylow* p*-subgroups of* G *are hypercentral.*

(ii) If G *satisfies* Min-n *, then the Sylow* p*-subgroups of* G *are hypercentral for every prime* p *.*

Proof. *(i)* Assume G is locally finite and satisfies p-Min-n . Let P be a Sylow p-subgroup of G and let $Q = O_{p'}(G)$. Then $P \cap Q = 1$, so that

$$P \cong P/P\cap Q \cong PQ/Q .$$

Hence we may assume that $Q = 1$. This implies that N is a p-group, so $N \leq P$. However G/N is locally nilpotent and so has a unique normal Sylow p-subgroup, which must obviously be P/N . Hence P is a normal p-subgroup of G and thus coincides with $O_p(G) = O_{p'p}(G)$. But $O_{p'p}(G) \cong O_{p'p}(G)/O_{p'}(G)$, and this is hypercentral by Theorem 3.14. Therefore P is hypercentral.

(ii) Assume that G satisfies Min-n . Then G/N is a locally nilpotent group satisfying Min-n , so it is an extension of a radicable abelian group by a finite nilpotent group. (See Robinson [33], Theorem 4.33.) Hence in particular G/N is soluble. Also N is contained in the Hirsch-Plotkin radical of G , so it is hypercentral by Theorem 3.14. However it is a consequence of Grün's Lemma (see Kurosh [20], p. 227) that the (transfinite) derived series of every hypercentral group reaches the unit subgroup. Therefore $N^{(\alpha)} = 1$ for some ordinal α . But the derived series of N cannot have an infinite number of strict inclusions, as its members are normal subgroups of G and G satisfies Min-n . Hence we may take $\alpha < \omega$, so that N is soluble. Therefore G is soluble, and

Theorem 3.15 now shows that G is locally finite. Furthermore G satisfies p-Min-n for every prime p, so part *(i)* shows that the Sylow p-subgroups of G are hypercentral, for each prime p. □

3.2 Quasi-radicability and Hypercentral Groups

The concept of quasi-radicability is very important for the study of soluble groups satisfying Min-n. One reason for this may be seen in the corollary to the next theorem, which shows that all soluble groups satisfying Min-n are 'almost' quasi-radicable groups satisfying Min-n.

3.21 THEOREM (Wilson [37]). *If a group satisfies* Min-n *then so does every subgroup of finite index.* □

3.22 COROLLARY. *Every soluble group satisfying* Min-n *has a quasi-radicable normal subgroup of finite index which also satisfies* Min-n.

Proof of Corollary. Let G be a soluble group satisfying Min-n. By Lemma 2.35, G has an $\underline{\underline{F}}$-perfect normal subgroup H of finite index. Theorem 3.21 shows that H satisfies Min-n, and H is quasi-radicable, since for soluble groups this is equivalent to being $\underline{\underline{F}}$-perfect (see Lemma 2.34). □

The results we aim to prove concerning the Sylow p-subgroups of metanilpotent groups satisfying Min-n rely ultimately on the interplay between the conditions of hypercentrality and quasi-radicability, which will concern us for the rest of this section.

The following result of Černikov is fundamental. (A proof may be found in Kurosh [20], §65.)

3.23 THEOREM (Černikov [5]). *If a periodic hypercentral group is quasi-radicable, then it is abelian.* □

We shall prove an extension of this theorem which will be useful for deriving the results we require. The key step is the following lemma, which has a number of useful corollaries.

3.24 LEMMA. *Let* H *be a periodic subgroup of a group* G . *If* R/S *is a quasi-radicable section of* G *such that* H *is normalized by* R *and centralized by* S , *then*

$$[H, R] = [H, R, HR] .$$

In particular, if either $H \leq R$ *or* H *is abelian, then*

$$[H, R] = [H, R, R] .$$

Proof. Let $h \in H$ and $x \in R$, and suppose h has order n .

We express each of the commutators $[h, x^n]$ and $[h^n, x]$ as a product of conjugates of $[h, x]$, using the well-known commutator identities, and obtain

$$\begin{aligned}[h, x^n] &= [h, x][h, x]^x \ldots [h, x]^{x^{n-1}} \\ &\equiv [h, x]^n \bmod [H, R, R]\end{aligned}$$

and

$$\begin{aligned}[h^n, x] &= [h, x]^{h^{n-1}} [h, x]^{h^{n-2}} \ldots [h, x] \\ &\equiv [h, x]^n \bmod [H, R, H] .\end{aligned}$$

From this we deduce that

$$[h, x^n] \equiv [h, x]^n \equiv [h^n, x] = 1 \bmod [H, R, HR] ,$$

that is,

$$[h, x^n] \in [H, R, HR] .$$

However, the commutators of the form $[h, x^n]$, with $h \in H$ and $x \in R$, generate $[H, R^n]$ modulo $[H, R, HR]$. Since R/S is quasi-radicable and $[H, S] = 1$ we have

$$[H, R] = [H, R^n S]$$
$$= [H, R^n] ,$$

and consequently

$$[H, R] = [H, R, HR]$$

as required. The remaining statement of the lemma is an immediate consequence of this. □

3.25 COROLLARY. *If a group G is an extension of a periodic group N by a quasi-radicable group, and if A is a normal subgroup of G contained in the centre of N, then*

$$[A, G, G] = [A, G] .$$

Proof. This is the special case of Lemma 3.24 obtained by replacing H, R and S by A, G and N respectively. □

Another special case of this lemma yields the following well-known result (see Robinson [33], Lemma 2.32).

3.26 COROLLARY. *A quasi-radicable subgroup of a periodic nilpotent group is contained in the centre.*

Proof. Let G be a periodic nilpotent group and A a quasi-radicable subgroup. Replacing H, R, S in Lemma 3.24 by G, A, 1 respectively, we have

$$[G, A] = [G, A, G]$$

and so

$$[A, G] = [A, G, G] = [A, G, G, G] = \ldots .$$

Because G is nilpotent, we conclude that $[A, G] = 1$. Hence A is contained in the centre. □

Another direct application of this lemma yields the following result, which we shall need later.

3.27 COROLLARY. *In a periodic quasi-radicable group the centre coincides with the hypercentre.*

Proof. Let G be a periodic quasi-radicable group. Replacing H, R, S in Lemma 3.24 by $\zeta_2(G)$, G, 1 respectively, we obtain

$$[\zeta_2(G), G] = [\zeta_2(G), G, G] = 1 ,$$

and therefore $\zeta_2(G) = \zeta_1(G)$. Thus the upper central series of G terminates at $\zeta_1(G)$, so this must be the hypercentre. □

We are now in a position to prove the following generalization of Černikov's theorem.

3.28 THEOREM. *Let G be a periodic hypercentral group having a normal subgroup N such that G/N is quasi-radicable. If N is nilpotent of class $c > 0$, then G is also nilpotent of class c.*

If in addition G satisfies Min-c *then G is a central product of the subgroup N and a radicable abelian group.*

Proof. We prove the first part of the theorem by induction on c.

Firstly suppose that $c = 1$. In this case we apply Corollary 3.25, taking for A the subgroup $N \cap \zeta_2(G)$. We conclude that

$$[N\cap\zeta_2(G), G] = [N\cap\zeta_2(G), G, G]$$
$$= 1 .$$

Hence

$$N \cap \zeta_2(G) \leq \zeta_1(G) .$$

This implies that

$$N.\zeta_1(G) \cap \zeta_2(G) = \zeta_1(G)$$

so that $N.\zeta_1(G)/\zeta_1(G)$ is a normal subgroup of $G/\zeta_1(G)$ which intersects the centre trivially. As $G/\zeta_1(G)$ is hypercentral this implies that $N \leq \zeta_1(G)$, by a well-known property of hypercentral groups (see e.g. Robinson [33], Lemma 1.51).

Now apply Lemma 3.24, taking for H the whole group G and for R/S the factor-group G/N. The conclusion is that

$$[G, G] = [G, G, G] .$$

However, by Theorem 3.23, G/N is abelian, so $G' \leq N$, and therefore

$$G' = [G', G] \leq [N, G] = 1 .$$

Hence G is abelian, concluding the case $c = 1$.

Now suppose that $c > 1$. By induction we may assume that $G/\gamma_c(N)$ is nilpotent of class $c - 1$: therefore $\gamma_c(G) = \gamma_c(N)$. This implies that

$$\gamma_c(G) \leq \zeta_1(N)$$

and hence we may apply Corollary 3.25 again, this time taking A to be $\gamma_c(G) \cap \zeta_2(G)$. We obtain

$$\begin{aligned}[\gamma_c(G)\cap\zeta_2(G), G] &= [\gamma_c(G)\cap\zeta_2(G), G, G] \\ &= 1\end{aligned}$$

so that $\gamma_c(G) \cap \zeta_2(G) \leq \zeta_1(G)$. By the argument used above for the case $c = 1$, we infer from this that $\gamma_c(G) \leq \zeta_1(G)$. Hence G is nilpotent of class c. This completes the induction argument, so the first part of the theorem is proved.

Suppose next that G satisfies Min-c. The series

$$G \geq G^{2!} \geq G^{3!} \geq G^{4!} \geq \dots$$

is a characteristic series of G, and so terminates after a finite number of steps at a characteristic subgroup $A = G^{n!}$, say. The factor-group G/A has finite exponent and A is quasi-radicable. As G/N is quasi-radicable, none of its proper factor-groups can have finite exponent: therefore $G = NA$. We have already shown that G is nilpotent, so the quasi-radicable subgroup A is central, by Corollary 3.26. Hence G is a central product of N and A.

This completes the proof. □

3.3 Sylow p-subgroups

In this section we put together the results of the two preceding sections to obtain an analogue for metanilpotent groups of the following theorem of McDougall.

3.31 THEOREM (McDougall [24]). *The Sylow p-subgroups of a quasi-radicable metabelian group satisfying* Min-n *are abelian, for each prime p.* □

Before proceeding with our discussion of metanilpotent groups satisfying Min-n, we first mention the following result of Baer, dealing with the structure of nilpotent groups satisfying Min-n.

3.32 THEOREM (Baer [1]). *Every nilpotent group satisfying* Min-n *is a central extension of a radicable abelian group satisfying* Min *by a finite nilpotent group.* □

To obtain our generalization of Theorem 3.31 we begin, as in Section 3.1, by considering the case of a locally finite metanilpotent group which satisfies p-Min-n for some particular prime p.

3.33 THEOREM. *Let G be a locally finite group having a nilpotent normal subgroup N of class $c > 0$ such that G/N is also nilpotent. Suppose that G satisfies p-Min-n, for some prime p, and let P be a Sylow p-subgroup of G.*

Then P is a finite extension of a nilpotent group P_0 of class at most c. Moreover P_0 is a central product of $P_0 \cap N$ and an abelian group.

If in addition the group G is p-quasi-radicable, then we may take $P_0 = P$.

Proof. Assume first that N is a p-group, so that $N \leq P$. Then P/N is the unique Sylow p-subgroup of G/N and is a direct factor of G/N. Since G/N satisfies p-Min-n, it follows that P/N satisfies Min-n. Now by Theorem 3.32 a nilpotent group satisfying Min-n is a finite extension of a radicable abelian group satisfying Min. Therefore P has a normal subgroup P_0 of finite index such that P_0/N is a radicable abelian group satisfying Min. Moreover if G is p-quasi-radicable, then P/N is quasi-radicable, so we have $P_0 = P$ in this case.

As P/N is a direct factor of G/N, the subgroup P_0 is normal in G and so satisfies Min-c. By Theorem 3.16, P is hypercentral and therefore so is P_0. Thus P_0 satisfies the conditions of Theorem 3.28. Consequently P_0 is nilpotent of class c and is a central product of N and a radicable abelian group. This completes the proof of the case where N is a p-group.

Next suppose that N is not a p-group. Let Q be the unique Sylow p'-subgroup of N: then Q is a normal subgroup of G. We write $\bar{G} = G/Q$, and for each subgroup H of G we let $\bar{H} = HQ/Q$. Since $P \cap Q = 1$ we have $P \cong \bar{P}$. Under this isomorphism $P \cap N$ is mapped onto $\overline{PQ} \cap \bar{N} = \bar{N}$. Now $\bar{P}$ is contained in the unique Sylow p-subgroup, $\bar{S}$ say, of $\bar{G}$. By the first part of the proof $\bar{S}$ has a subgroup $\bar{S}_0$ of finite index which is expressible as a central product

$$\bar{S}_0 = \bar{N}\bar{A}\ . \qquad (1)$$

If G is p-quasi-radicable then so is $\bar{G}$ and we may take $\bar{S}_0 = \bar{S}$ in this case. From (1) we have

$$\bar{S}_0 \cap \bar{P} = \bar{N}\bar{A} \cap \bar{P}$$

$$= \bar{N}(\bar{A}\cap\bar{P}) \ , \qquad (2)$$

so that $\bar{S}_0 \cap \bar{P}$ is a central product of $\bar{N}$ and an abelian group. Let P_0 be the subgroup corresponding to $\bar{S}_0 \cap \bar{P}$ under the above isomorphism between P and $\bar{P}$. Then P_0 has finite index in P because

$$|\bar{P} : \bar{S}_0\cap\bar{P}| \leq |\bar{S} : \bar{S}_0| \ ,$$

and if G is p-quasi-radicable then $\bar{S}_0 \cap \bar{P} = \bar{P}$ and so $P_0 = P$ in this case. Using the isomorphism between P and $\bar{P}$ we deduce from (2) that P_0 is a central product of $N \cap P$ and an abelian group. Thus we have established the theorem in both cases. □

We can now easily deduce the promised analogue of Theorem 3.31.

3.34 THEOREM. *Let G be a quasi-radicable group satisfying* Min-n *and having a nilpotent normal subgroup N of class $c > 0$ such that G/N is also nilpotent. Then the Sylow p-subgroups of G are nilpotent of class at most c, for each prime p, and each one is a central product of a Sylow p-subgroup of N and an abelian group.*

Proof. By Theorem 3.15, G is locally finite. Also G satisfies p-Min-n and is p-quasi-radicable for every prime p. The result therefore follows immediately from Theorem 3.33. □

3.35 REMARK. In fact any group satisfying the conditions of Theorem 3.34 will be nilpotent-by-abelian. This is because a nilpotent homomorphic image of such a group is both periodic and quasi-radicable, and hence is abelian by Corollary 3.26.

We conclude this section by using Theorem 3.34 to obtain some further information on the Hirsch-Plotkin radical of a quasi-radicable nilpotent-by-abelian group satisfying Min-n.

3.36 LEMMA. *Let G be a quasi-radicable nilpotent-by-abelian group satisfying* Min-n *and suppose G' has nilpotency class c. Then the Hirsch-Plotkin radical $\rho(G)$ is nilpotent of class c, and is a central product of G' and an abelian group.*

Proof. Let $R = \rho(G)$. Since G' is nilpotent, we have $G' \leq R$. Now R is locally nilpotent, so we can write

$$R = \mathrm{Dr}_p\, R_p ,$$

where R_p is the unique Sylow p-subgroup of R for every prime p. Choose a Sylow p-subgroup S_p of G for each prime p: then we have

$$G'_p = G' \cap S_p \leq R_p \leq S_p ,$$

for each p, where G'_p is the Sylow p-subgroup of G'. By Theorem 3.34, S_p is a central product

$$S_p = (G' \cap S_p)A_p = G'_p A_p$$

for some abelian subgroup A_p of G. Therefore R_p is a central product

$$R_p = G'_p(A_p \cap R_p) .$$

Consequently if we set

$$A = \mathrm{Dr}_p\, (A_p \cap R_p)$$

then R is a central product

$$R = G'A$$

as claimed. From this it follows immediately that R is nilpotent of class c. □

NOTE. When we have proved Theorem 3.43, we shall be able to show that if a group G satisfies the hypotheses of Lemma 3.36 then we have in fact

$$\rho(G) = G'.\zeta(G) .$$

3.4 A Splitting Theorem

We now aim to generalize the following result of McDougall.

3.41 THEOREM (McDougall [24]). *Every quasi-radicable metabelian group satisfying* Min-n *splits over its derived group.* □

As we have noted in Remark 3.35, every quasi-radicable metanilpotent group satisfying Min-n is in fact nilpotent-by-abelian. However, if the word 'metabelian' in Theorem 3.41 is replaced by 'nilpotent-by-abelian', or even 'centre-by-metabelian', then the resulting statement is no longer true. This is shown by an example in [24] (Example 3). Therefore we cannot hope to prove the obvious analogue of Theorem 3.41.

Now the example mentioned above is constructed by taking a quasi-radicable centre-by-metabelian group satisfying Min-n which does split over its derived group and forming a central product of this group and a quasicyclic group in such a way that the derived group is no longer complemented in the larger group. The theorem we shall prove implies that all quasi-radicable nilpotent-by-abelian groups satisfying Min-n which do not split over their derived groups must be constructed in a manner similar to this. In particular we prove that if the centre of a quasi-radicable nilpotent-by-abelian group satisfying Min-n intersects the derived group trivially then the group does split over its derived group.

Before we come to this theorem, we need to note the following consequence of Theorem 3.41.

3.42 LEMMA (McDougall). *Let* G *be a quasi-radicable* *metabelian group satisfying* Min-n . *Then* $\rho(G) = G'.\zeta(G)$.

Proof. Let $R = \rho(G)$. Because G' is abelian, we have $G' \leq R$. By Theorem 3.41, G' has a complement K in G . Hence

$$R = G'(K \cap R) . \tag{1}$$

Now K is abelian, being isomorphic to G/G' . Also R is abelian, by Lemma 3.36, so $[K \cap R, G] = [K \cap R, G'K] = 1$. Hence (1) shows that $R \leq G'.\zeta(G)$.

However, since $G'.\zeta(G)$ is an abelian normal subgroup of G , the reverse inclusion also holds. Therefore $R = G'.\zeta(G)$. □

3.43 THEOREM. *Let* G *be a quasi-radicable nilpotent-by-abelian group satisfying* Min-n . *Then* G *has a radicable abelian subgroup* K *satisfying* Min *such that* $G = G'K$ *and* $G' \cap K$ *is contained in the centre of* G .

Proof. We use induction on the class c of G' .

If $c \leq 1$, then G is metabelian and by Theorem 3.41, G' has a complement K in G . Since $K \cong G/G'$, it is clear that K is a radicable abelian group satisfying Min .

Now suppose $c > 1$. We deal first with the case where $G' \cap \zeta(G) = 1$. In this case we have to show that G' has a complement in G .

Let $A = \zeta(G')$. The group G/A satisfies the hypotheses of the theorem and its derived group G'/A has class $c - 1$. Therefore by induction G/A has a radicable abelian subgroup M/A satisfying Min such that

$$G/A = (G'/A)(M/A)$$

and

$$(G'/A) \cap (M/A) \leq \zeta(G/A) .$$

Writing $Z/A = \zeta(G/A)$ we therefore have

$$G = G'M$$

and

$$G' \cap M \leq Z \ .$$

We claim that M is a metabelian group satisfying Min-n. Firstly, M is metabelian, for both A and M/A are abelian. Further, since A is the centre of G', any normal subgroup of M lying inside A is normal in G' and therefore also in $G = G'M$. Hence A satisfies Min-M. But M/A satisfies Min, so we conclude that M satisfies Min-n.

By Corollary 3.22, M is a finite extension of a quasi-radicable metabelian group L satisfying Min-n. By the case $c = 1$ considered above, L' is complemented in L by a radicable abelian group K satisfying Min. We show that K is the complement for G' we are seeking.

Firstly, because M/L is finite and M/A is radicable, we have

$$M = LA \ .$$

Hence

$$M' = L'[L, A] \leq L \ .$$

Thus M' is a normal abelian subgroup of L and therefore is contained in the Hirsch-Plotkin radical $\rho(L)$. By Lemma 3.42 we have

$$\rho(L) = L'.\zeta(L)$$

so we can write

$$M' = L'\big(M' \cap \zeta(L)\big) \ .$$

However

$$\begin{aligned} M' \cap \zeta(L) &\leq A \cap \zeta(L) \\ &= \zeta(G') \cap \zeta(L) \\ &\leq G' \cap \zeta(G) = 1 \end{aligned}$$

by our assumption. We conclude that $M' = L'$.

Hence K is a complement to M' in L : that is $L = M'K$ and $M' \cap K = 1$. Therefore

$$\begin{aligned} G &= G'M \\ &= G'AL = G'L \\ &= G'M'K = G'K . \end{aligned}$$

It now remains only to show that $G' \cap K = 1$.

Since G' is nilpotent and K is abelian, we can express each as a direct product

$$G' = \underset{p}{\mathrm{Dr}}\, G'_p ,$$

$$K = \underset{p}{\mathrm{Dr}}\, K_p ,$$

where G'_p and K_p are the unique Sylow p-subgroups of G' and K respectively, for each prime p . Now for each p the subgroup G'_pK_p is a p-subgroup of G and so by Theorem 3.34 it is nilpotent. But K is radicable and hence so is each of the direct factors K_p . It therefore follows from Corollary 3.26 that

$$\left[G'_p, K_p\right] = 1$$

and so

$$\left[G'_p, (G' \cap K)_p\right] = 1$$

for each prime p (where $(G' \cap K)_p$ denotes the Sylow p-subgroup of $G' \cap K$). Also if p and q are distinct primes then

$$\left[G'_p, (G' \cap K)_q\right] = 1$$

since G'_p and $(G' \cap K)_q$ lie in distinct direct factors of G' .

Consequently

$$\begin{aligned} [G', G' \cap K] &= \left[\underset{p}{\mathrm{Dr}}\, G'_p, \underset{p}{\mathrm{Dr}}\, (G' \cap K)_p\right] \\ &= 1 . \end{aligned}$$

Because $G = G'K$ and K is abelian, we deduce from this that

$$\begin{aligned}[G, G' \cap K] &= [G'K, G' \cap K] \\ &= [G', G' \cap K] \\ &= 1 .\end{aligned}$$

Hence $G' \cap K \leq G' \cap \zeta(G) = 1$, and K is the complement required. This completes the inductive step in the case where $G' \cap \zeta(G) = 1$.

Suppose next that $G' \cap \zeta(G) \neq 1$ and write $\bar{G} = G/\{G' \cap \zeta(G)\}$. If H is the subgroup of G defined by

$$H/\{G' \cap \zeta(G)\} = \bar{G}' \cap \zeta(\bar{G})$$

then we have $H \leq G' \cap \zeta_2(G)$. However by Corollary 3.27, $\zeta_1(G)$ coincides with the hypercentre of G . Therefore $H \leq G' \cap \zeta_1(G)$ and consequently the centre and the derived group of $\bar{G}$ intersect trivially.

Applying the first part of the proof to the group $\bar{G}$, we see that $\bar{G}$ has a radicable abelian subgroup $\bar{K}_1$ satisfying Min which complements $\bar{G}'$. If $\bar{K}_1 = K_1/\{G' \cap \zeta(G)\}$ then

$$K_1' \leq G' \cap \zeta(G) \cap K_1 \leq \zeta(K_1)$$

so K_1 is nilpotent. Also $G' \cap \zeta(G)$ satisfies Min , being a central subgroup of G , and it follows that K_1 satisfies Min . Hence by Theorem 3.32, K_1 has a radicable abelian subgroup K of finite index lying in its centre. Because $K_1/\{G' \cap \zeta(G)\}$ is quasi-radicable and K_1/K is finite we have

$$K_1 = \{G' \cap \zeta(G)\}K .$$

Therefore

$$G = G'K_1 = G'K ,$$

and

$$G' \cap K \leq G' \cap K_1 \leq \zeta(G) .$$

Thus K has the properties claimed in this case too, and so by induction we obtain the required result. □

We note as a corollary the following result, which is immediate from the above proof.

3.44 COROLLARY. *Let G be a quasi-radicable nilpotent-by-abelian group satisfying* Min-n . *If $G' \cap \zeta(G) = 1$ then G splits over G' .* □

By invoking Theorem 3.43 we can now obtain an improved version of Lemma 3.36.

3.45 THEOREM. *Let G be a quasi-radicable nilpotent-by-abelian group satisfying* Min-n . *Then* $\rho(G) = G'.\zeta(G)$.

Proof. Let $R = \rho(G)$. Since G' is nilpotent we have $G' \leq R$. Also Theorem 3.43 shows that there is a radicable abelian subgroup K such that $G = G'K$.

Now G' and K have direct decompositions

$$G' = \operatorname{Dr}_p G'_p ,$$

$$K = \operatorname{Dr}_p K_p ,$$

where G'_p and K_p are the unique Sylow p-subgroups of G' and K respectively, for each prime p . Each of the p-subgroups G'_pK_p is nilpotent, by Theorem 3.34. Also, since K is radicable, so is K_p for every p , and we deduce from Corollary 3.26 that

$$[G'_p, K_p] = 1 . \qquad (1)$$

Now $R = G'(K \cap R)$, and as R is locally nilpotent it is also the direct product of its Sylow subgroups. If we write A_p for the Sylow p-subgroup of $K \cap R$, for each prime p , then G'_pA_p is clearly the Sylow p-subgroup of R ; hence

$$R = \mathrm{Dr}_p\, G'_p A_p \ .$$

By (1) we have $[G'_p, A_p] = 1$ for each p and hence also

$$[G', K \cap R] = 1 \ .$$

As K is abelian it follows that

$$\begin{aligned}[G, K\cap R] &= [G'K, K\cap R] \\ &= [G', K\cap R] \\ &= 1 \ .\end{aligned}$$

Therefore $K \cap R \leq \zeta(G)$, and consequently

$$R = G'(K\cap R) \leq G'.\zeta(G) \ .$$

However, the reverse inclusion also holds, since $G'.\zeta(G)$ is a normal nilpotent subgroup of G . Hence we have

$$\rho(G) = R = G'.\zeta(G)$$

as claimed. □

3.5 Countability

We are now easily able to generalize the following result of McDougall.

3.51 THEOREM (McDougall [24]). *Every metabelian group satisfying* Min-n *is countable.* □

In this case the analogue for metanilpotent groups is the obvious one.

3.52 THEOREM. *Every metanilpotent group satisfying* Min-n *is countable.*

Proof. Let G be a metanilpotent group satisfying Min-n . By Corollary 3.22, G has a quasi-radicable subgroup of finite index satisfying Min-n , and we may clearly suppose for the purposes of the proof that G itself is quasi-radicable. Hence G is nilpotent-by-abelian (see Remark 3.35).

By Theorem 3.43, G has a radicable abelian subgroup K satisfying Min such that

$$G = G'K \quad \text{and} \quad G' \cap K \leq \zeta(G) .$$

Suppose G' has nilpotency class c and write

$$A_i = \gamma_i(G')/\gamma_{i+1}(G')$$

for $1 \leq i \leq c$. The elements of K induce automorphisms on the factors A_i and using this action we can form the split extensions

$$H_i = A_i \rtimes K$$

for $1 \leq i \leq c$. Since A_i is a central factor of G' for each i, the K-invariant subgroups of G' lying between $\gamma_{i+1}(G')$ and $\gamma_i(G')$ are normal in G', and hence also in $G = G'K$. Thus the groups H_i are all metabelian groups satisfying Min-n; consequently they are countable, by Theorem 3.51. From this it follows at once that G is countable. □

3.6 Some Examples

In this section we describe a method of constructing examples of quasi-radicable nilpotent-by-abelian groups satisfying Min-n. Using this method we obtain examples of such groups with derived groups of arbitrarily large nilpotency class and also examples having arbitrarily large derived lengths. The construction is based on that used by Čarin [3] for establishing the existence of metabelian groups satisfying Min-n but not Min. We therefore begin by describing Čarin's construction briefly.

3.61 Čarin's construction. Let p and q be distinct primes and let K denote the algebraic closure of the field $\mathrm{GF}(p)$. The multiplicative group K^* of K is a direct product of quasicyclic groups and in particular has a subgroup Q of type C_{q^∞}. Let F be the subfield of K generated by the elements of Q. Multiplication by any element of Q induces an automorphism on the additive group F^+ of F, and since F^+ is generated by the elements of Q, the group F^+ is irreducible under this action of Q. We let G be the split extension of F^+ by Q, the elements of Q transforming F^+ according to this multiplicative action.

The group G is metabelian, being an extension of an infinite elementary abelian p-group by a group of type C_{q^∞}, and the subgroup F^+ is a unique minimal normal subgroup of G and so coincides with G'. The factor-group G/G' is isomorphic to Q and so is radicable and satisfies Min : hence G is quasi-radicable and satisfies Min-n.

The group G has a faithful representation as a subgroup of the linear group $\mathrm{GL}(2, F)$, for the matrices

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \quad (\alpha \in F)$$

form a subgroup of $\mathrm{GL}(2, F)$ isomorphic to F^+ and the matrices

$$\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \quad (\omega \in Q)$$

form a subgroup isomorphic to Q, and we have

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \omega\alpha & 1 \end{pmatrix} .$$

Hence these isomorphisms may be extended to give an isomorphism between G and the group of all matrices

$$\begin{pmatrix} \omega & 0 \\ \alpha & 1 \end{pmatrix} \quad (\omega \in Q,\ \alpha \in F)$$

in $GL(2, F)$.

3.62 The General Construction. Motivated by this representation, we can now construct nilpotent-by-abelian groups with similar properties.

Keeping the notation of 3.61, let n be any positive integer with $2 \le n \le q$, and let N be the group of lower unitriangular $n \times n$ matrices over F . If we write e_{ij} for the $n \times n$ matrix having 1 in the (i, j) place and 0 elsewhere, then N consists of all sums

$$1 + \sum \alpha_{ij} e_{ij} \quad (\alpha_{ij} \in F) \tag{1}$$

where the summation extends over all pairs of integers i, j with

$$1 \le j < i \le n . \tag{2}$$

It is well-known and easy to verify that N is nilpotent of class $n - 1$, the k-th term of its lower central series consisting (when $1 \le k \le n$) of all sums (1) with $\alpha_{ij} = 0$ when $i - j \le k$. (That is, $\gamma_k(N)$ consists of all matrices in N with zeros on the first $k - 1$ subdiagonals.)

Now let $\bar{Q}$ be the group of all $n \times n$ diagonal matrices of the form

$$\omega^{n-1} e_{11} + \omega^{n-2} e_{22} + \dots + e_{nn} \tag{3}$$

where $\omega \in Q$. The mapping assigning to the element (3) of $\bar{Q}$ the element ω of Q is evidently an isomorphism between $\bar{Q}$ and Q .

We take G to be the group generated by N and $\bar{Q}$. On conjugation by an element (3) of $\bar{Q}$, the element (1) of N is mapped to the element

$$1 + \sum \alpha_{ij}\omega^{i-j}e_{ij} ,$$

as is easily verified by direct computation. Consequently, for each pair i, j of integers satisfying (2) the subgroup

$$A_{ij} = \{1+\alpha e_{ij} : \alpha \in F\}$$

is invariant under conjugation by elements of $\bar{Q}$. As the mapping $1 + \alpha e_{ij} \mapsto \alpha$ is clearly an isomorphism from A_{ij} onto F^+, the subgroup $A_{ij}\bar{Q}$ of G is isomorphic to the split extension of F^+ by Q with each element $\omega \in Q$ transforming F^+ according to the rule

$$\alpha \mapsto \omega^{i-j}\alpha \quad (\alpha \in F) .$$

However, since $1 \leq i-j \leq q-1$, the mapping $\omega \mapsto \omega^{i-j}$ is an automorphism of Q; hence $A_{ij}\bar{Q}$ is isomorphic to one of the groups constructed in 3.61. Hence A_{ij} is transformed irreducibly by $\bar{Q}$.

Now by the remarks made earlier $\gamma_k(N)$ is generated for each k by the subgroups A_{ij} with $i-j \leq k$, and it is a simple matter to verify that

$$\gamma_k(N)/\gamma_{k+1}(N) = \bar{A}_{k+1,1} \times \bar{A}_{k+2,2} \times \ldots \times \bar{A}_{n,n-k} ,$$

where the bars denote the images of the subgroups A_{ij} under the canonical projection $N \to N/\gamma_{k+1}(N)$. For each i, the factor $\bar{A}_{k+i,i}$ is transformed irreducibly by $\bar{Q}$. Thus each lower central factor $\gamma_k(N)/\gamma_{k+1}(N)$ is a direct product of a finite number of subgroups, each irreducible under the action of $\bar{Q}$. Hence the lower central series of N can be refined to a finite G-chief series between 1 and N, so N certainly satisfies Min-G. But

$G/N \cong Q$, which is a group of type C_{q^∞} , so it follows that G is a quasi-radicable group satisfying Min-n .

We have therefore constructed a quasi-radicable group which satisfies Min-n and is an extension of a nilpotent group of class $n - 1$ by an abelian group. Since n was chosen to be any positive integer with $2 \le n \le q$, and since q was any prime not equal to p , this construction gives us examples of quasi-radicable nilpotent-by-abelian groups satisfying Min-n and having derived groups with any prescribed nilpotency class. Moreover since a group of $n \times n$ unitriangular matrices (with $n > 1$) has derived length $[\log_2(n-1)] + 1$ (see e.g. Robinson [33], p. 26), these examples also include groups of arbitrarily large derived lengths.

CHAPTER FOUR

TWISTED WREATH PRODUCTS AND METABELIAN GROUPS SATISFYING MIN-N

In this chapter our object is to prove a result on the structure of quasi-radicable metabelian groups satisfying Min-n of a rather different kind from the results of McDougall which we considered in the last chapter. One of McDougall's results (Theorem 3.41) shows that every group in this class is a split extension of its derived group by a radicable abelian group satisfying Min . We now investigate the nature of these split extensions, and indicate a connection between this class of groups and the class of metabelian groups satisfying Min .

The connection between these classes is described using the concept of a twisted wreath product. The twisted wreath products that occur in the description are all 'close' to ordinary wreath products, since the subgroup which does the 'twisting' turns out in each case to be a finite group. We shall show that some particular types of quasi-radicable metabelian groups satisfying Min-n are themselves expressible as twisted wreath products of this type. However we also obtain a rather more complicated description which is valid for arbitrary quasi-radicable metabelian groups satisfying Min-n . We state below the general theorem which we intend to prove.

4.41 THEOREM. *Let G be a quasi-radicable metabelian group satisfying* Min-n *and let A be a complement to G' in G . Then G' is expressible as a direct product*

$$G' = V_1 \times \ldots \times V_n$$

of a finite number of normal subgroups $V_1, \ldots, V_n$ of G such that, for $i = 1, 2, \ldots, n$, the group

$$G_i = V_i A$$

has the following structure:

$$G_i = \left(U_i \operatorname{wr}_{F_i} A_i\right) \times B_i$$

where U_i, F_i, A_i *and* B_i *are subgroups such that*

(i) $A = A_i \times B_i$,

(ii) F_i *is finite,*

(iii) $U_i F_i \times B_i$ *is a group satisfying* Min .

Each twisted wreath product $U_i \operatorname{wr}_{F_i} A_i$ occurring in the statement of this theorem has the property that the split extension $U_i F_i$ and the group A_i are both groups satisfying Min , while the twisted wreath product itself satisfies Min-n (but not in general Min). Thus we see that it is sometimes possible to use a twisted wreath product to construct a metabelian group satisfying Min-n from two groups satisfying Min in a manner somewhat analogous to the process of constructing a group satisfying Max-n using a wreath product of two groups satisfying Max . Unfortunately it appears to be a difficult problem to determine the conditions under which this method can actually be used for the construction of examples, and we have made little progress with this.

To prove Theorem 4.41 we first introduce in section 4.1 the concept of a system of imprimitivity for a normal subgroup of a group. This is closely related to the systems of imprimitivity studied in connection with both linear groups and permutation groups. Using this concept we obtain a simple criterion for a group to be expressible as a twisted wreath product of certain subgroups. Then we transform the problem into one about modules in a standard way, and in sections 4.2 and 4.3 we apply some results from Hartley and McDougall [12] to analyse the structure of the modules which occur. Finally in section 4.4 we use the properties of these modules to prove the above theorem.

4.1 Systems of Imprimitivity and Twisted Wreath Products

We begin this section by defining systems of imprimitivity, both for modules and for normal subgroups of a group.

4.11 DEFINITION. Let G be a group and K a commutative ring with identity, and let V be a KG-module. A set

$$\{U_\lambda : \lambda \in \Lambda\}$$

of K-submodules of V is said to be a *system of imprimitivity* for V if

(i) V is a K-admissible direct sum

$$V = \bigoplus_{\lambda\in\Lambda} U_\lambda ,$$

(ii) multiplication by elements of G permutes the submodules U_λ among themselves: that is, if $g \in G$ and $\lambda \in \Lambda$, then

$$U_\lambda g = U_{\lambda'}$$

for some $\lambda' \in \Lambda$.

The module V is said to be *primitive* if its only system of imprimitivity is the set $\{V\}$; otherwise V is said to be *imprimitive*.

4.12 DEFINITION. Let N be a normal subgroup of a group G. A set

$$\{M_\lambda : \lambda \in \Lambda\}$$

of subgroups of N is said to be a *system of imprimitivity for* N

in G if

(i) $N = \operatorname{Dr}_{\lambda\in\Lambda} M_\lambda$,

(ii) conjugation by elements of G permutes the subgroups M_λ among themselves: that is, if $g \in G$ and $\lambda \in \Lambda$, then $M_\lambda^g = M_{\lambda'}$ for some $\lambda' \in \Lambda$.

The normal subgroup N is said to be a *primitive normal subgroup* of G if the only system of imprimitivity for N in G is the set $\{N\}$; otherwise N is said to be an *imprimitive normal subgroup* of G .

REMARKS. (1) If N is an abelian normal subgroup of a group G then we may view N also as a $\mathbb{Z}G$-module (where $\mathbb{Z}$ is the ring of integers), with the elements of G acting on N by conjugation. In this case the systems of imprimitivity for N as a normal subgroup clearly are also systems of imprimitivity for N as a $\mathbb{Z}G$-module, and conversely.

(2) Suppose a group G is a split extension

$$G = N \rtimes A$$

and let S be a system of imprimitivity for N in G . It follows easily from the definition that S must contain all the distinct conjugates in A of each of its members. Hence if $\{M_\sigma : \sigma \in \Sigma\}$ is a subset of S consisting of one representative from each conjugacy class in A of the members of S , and if T_σ is a transversal to $\mathrm{N}_A(M_\sigma)$ for each $\sigma \in \Sigma$, then S is a disjoint union

$$S = \bigcup_{\sigma\in\Sigma} \left\{M_\sigma^t : t \in T_\sigma\right\} . \tag{1}$$

Also for each $\sigma \in \Sigma$ the subgroup

$$N_\sigma = \operatorname{Dr}_{t\in T_\sigma} M_\sigma^t$$

is clearly normal in G , and N is a direct product

$$N = \operatorname{Dr}_{\sigma \in \Sigma} N_\sigma .$$

Moreover, for each $\sigma \in \Sigma$ the action of A on N_σ is determined by that of $\mathrm{N}_A(M_\sigma)$ on M_σ . This is because each $a \in A$ may be written in a unique way as a product $a = b_\sigma t_\sigma$, where $b_\sigma \in \mathrm{N}_A(M_\sigma)$ and $t_\sigma \in T_\sigma$, and the action of the elements of T_σ is clearly specified by the equation (1).

Thus by establishing the existence of a system of imprimitivity S of this form, we can describe the action of A on N in terms of the action of $\mathrm{N}_A(M_\sigma)$ on M_σ for every $\sigma \in \Sigma$.

Using the concept of a system of imprimitivity for a normal subgroup, one gets a convenient criterion for expressing a group as a twisted wreath product, which we describe in the following lemma.

4.13 LEMMA. *Let the group* G *be a split extension*

$$G = A \rbrack B$$

and suppose A *has a system of imprimitivity consisting of conjugates in* B *of some subgroup* $\bar{A}$. *Let* $\bar{B} = \mathrm{N}_B(\bar{A})$: *then* G *is isomorphic to the twisted wreath product*

$$W = \bar{A} \operatorname{wr}_{\bar{B}} B ,$$

where the action of $\bar{B}$ *on* $\bar{A}$ *in* W *is that induced by conjugation in the group* G .

Proof. Let T be a right transversal to $\bar{B}$ in B containing the identity element. The set $\{\bar{A}^t : t \in T\}$ is a complete set of conjugates of $\bar{A}$ in G . Since these conjugates form a system of imprimitivity for A , by hypothesis, we have

$$A = \operatorname{Dr}_{t \in T} \bar{A}^t .$$

Now the base group of the twisted wreath product W is a direct product of copies of $\bar{A}$ indexed by the elements of a transversal to $\bar{B}$ in B, and since the twisted wreath product is determined independently of the choice of transversal we may assume this transversal is T. If we identify $\bar{A}$ with the 1-component of the base group, then the base group is a direct product

$$\operatorname*{Dr}_{t \in T} \bar{A}^t$$

and so is isomorphic to A. The group W is a split extension of its base group by the group B, so to verify that W is isomorphic to G we need only check that the above isomorphism is compatible with the action of B. But this is straightforward. □

4.2 Systems of Imprimitivity for Irreducible Modules

We shall accomplish the main step in the proof of Theorem 4.41 by proving certain facts about the systems of imprimitivity of modules for radicable abelian groups satisfying Min. If G is a quasi-radicable metabelian group satisfying Min-n, then G' has a complement, A say, by Theorem 3.41. We may view G' in a natural way as an A-module, with the elements of A acting on G' by conjugation. The submodules of G' are then precisely the normal subgroups of G contained in G'; consequently G' will satisfy the minimal condition on A-submodules. We now study the systems of imprimitivity for A-modules of this type, beginning with the case where G' is an irreducible A-module.

Notation. The modules we consider all arise from abelian normal subgroups in the manner described above, and we shall accordingly use notation which emphasizes this fact.

Let G be a group and K a commutative ring with identity.

Suppose V is a KG-module and U is a K-submodule of V, not necessarily invariant under the action of G. We write

$$\mathrm{N}_G(U) = \{g \in G : Ug = U\}$$

and

$$\mathrm{C}_G(U) = \{g \in G : ug = u \text{ for all } u \in U\} ,$$

and refer to all K-modules of the form Ug, where $g \in G$, as *conjugates* of U.

If H is a subgroup of G then we may view V also as a KH-module, simply by restricting the action of G: when we wish to emphasize that V is to be considered as a KH-module in this way, we denote it by V_H.

If $K = \mathbb{Z}$, the ring of integers, we shall often refer to V as a *G-module* rather than a $\mathbb{Z}G$-module.

The underlying additive group of V will be denoted by V^+. For any positive integer n, we define

$$V[n] = \{v \in V : nv = 0\} .$$

If V is a G-module then we use the term *divisible hull of* V to mean the divisible hull of V^+ (in the sense of Fuchs [6], §24). Thus the divisible hull of a G-module is not necessarily a G-module, but only a divisible abelian group containing the module in general.

4.21. We now consider a particular class of irreducible modules for abelian groups. The discussion which follows is taken from Hartley and McDougall [12], p. 121.

Let A be a periodic abelian group and p a prime, and let θ be a homomorphism from A into the multiplicative group of the field $\mathrm{k} = \mathrm{GF}(p^\infty)$ (the algebraic closure of the field $\mathbb{Z}_p$ of p elements). Define a $\mathbb{Z}_pA$-module $K(\theta)$ as follows: the underlying vector space of $K(\theta)$ is the subfield of k generated by $A\theta$, viewed as a vector space over Z_p, and the action of an element $a \in A$ on $K(\theta)$ is given by

$$x.a = x(a\theta) \ , \ \{x \in K(\theta)\} \ ,$$

where the product on the right is according to the field multiplication in k. This action can be extended to Z_pA by linearity, and with this definition $K(\theta)$ becomes a Z_pA-module. Moreover, since the elements of $A\theta$ are roots of unity, they generate $K(\theta)$ as an additive group, and from this it easily follows that $K(\theta)$ is an irreducible Z_pA-module.

The next lemma is a straightforward extension of part (i) of Lemma 2.5 of Hartley and McDougall [12] (*cf.* also Satz 3.10, p. 165, of Huppert [19]). Before stating the lemma we introduce a piece of notation that we use throughout this chapter.

Notation. If a and b are relatively prime positive integers, then we write

$$\mathrm{ord}(a, b)$$

for the *order* of a modulo b; that is, the least positive integer n such that $a^n \equiv 1 \pmod b$.

4.22 LEMMA. *Let A be a periodic abelian group and p a prime, and let V be an irreducible Z_pA-module. Then*

(i) there is a homomorphism θ from A into the multiplicative group of $\mathsf{k} = \mathrm{GF}(p^\infty)$ such that

$$V \cong K(\theta) \ ,$$

(ii) $C_A(V) = \ker\theta$,

(iii) $A/C_A(V)$ is a locally cyclic p'-group,

(iv) if $A/C_A(V)$ has finite order m then the dimension of

V over Z_p *is* $\operatorname{ord}(p, m)$.

Proof. For *(i)* we refer to Hartley and McDougall [12], Lemma 2.5.

Part *(ii)* is immediate from *(i)* and the definition of $K(\theta)$.

To prove part *(iii)*, note that by *(ii)* we have

$$A/C_A(V) = A/\ker\theta \cong A\theta \leq k^* \tag{1}$$

and as k^* is a locally cyclic p'-group so is $A/C_A(V)$.

To prove part *(iv)*, suppose $|A/C_A(V)| = m$. Then using *(iii)* we see that $A\theta$ is a cyclic subgroup of k^* of order m . Consequently a generator of $A\theta$ is a primitive m-th root of unity in k , so the subfield of k generated by $A\theta$ must be isomorphic to the Galois field $GF(p^f)$, where f is the least positive integer for which the order $p^f - 1$ of the multiplicative group of $GF(p^f)$ is divisible by m . But this means that $f = \operatorname{ord}(p, m)$, so that $K(\theta)$, having its underlying vector space equal to $GF(p^f)$, has dimension $\operatorname{ord}(p, m)$ over Z_p . Therefore by *(i)* the dimension of V over Z_p is also $\operatorname{ord}(p, m)$. □

We next prove two lemmas which will provide us with a way of finding systems of imprimitivity for the modules we are studying.

4.23 LEMMA. *Let G be a group, H a subgroup of finite index and K an arbitrary field. Suppose V is a finite-dimensional KG-module and U is a KH-submodule of V_H such that $V = UG$. If*

$$\dim_K V = |G : H| \dim_K U \tag{1}$$

then the conjugates of U form a system of imprimitivity for V and $N_G(U) = H$.

Proof. Let T be a right transversal to H in G . Since

$V = UG$ we have

$$V = \sum_{t \in T} Ut \ . \tag{2}$$

Each of the subspaces Ut has dimension equal to that of U and the number of summands is $|T| = |G : H|$. Hence equation (1) implies that the sum in (2) is a direct sum.

If $t \in T$ and $g \in G$ then there is an element $t' \in T$ such that $tg(t')^{-1} \in H$. Hence

$$Utg = Ut'$$

and therefore multiplication by elements of G permutes the subspaces Ut . Hence $\{Ut : t \in T\}$ is a system of imprimitivity for V . It is clear from this that $\mathrm{N}_G(U) = H$. □

The next lemma is an extension of Lemma 4.23 to cope with situations where the index $|G : H|$ may be infinite.

4.24 LEMMA. *Let G be a group, H a subgroup and K an arbitrary field. Suppose V is a KG-module and U is a KH-submodule of V_H of finite dimension over K such that $V = UG$. If G has subgroups $G_1, G_2, G_3, \ldots$ with*

$$H = G_1 \leq G_2 \leq G_3 \leq \ldots ,$$

and

$$G = \bigcup_{n \geq 1} G_n ,$$

such that $|G_n : H|$ is finite for each $n \geq 1$, and if the equation

$$\dim UG_n = |G_n : H| \dim U \tag{1}$$

is valid for each $n \geq 1$, then the conjugates of U in G form a system of imprimitivity for V , and $\mathrm{N}_G(U) = H$.

Proof. Choose a right transversal T to H in G which is expressible as a union

$$T = \bigcup_{n\geq 1} T_n ,$$

where T_n is a transversal to H in G_n, for each n, and $T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots$. As in Lemma 4.23 we have

$$V = \sum_{t\in T} Ut . \tag{2}$$

Moreover equation (1) shows that, for each n, the sum

$$UG_n = \sum_{t\in T_n} Ut$$

is a direct sum. Hence so is the sum in (2) and the result follows as in Lemma 4.23. □

In our applications of Lemma 4.24 we shall make use of the fact, established in Lemma 4.22 *(iv)*, that the dimensions of certain irreducible modules for abelian groups are given by the values of the number-theoretic function $\operatorname{ord}(p, m)$. We now embark on a series of number-theoretic lemmas aimed at answering the following question: if p is a prime and π is a finite set of primes not containing p, how do the values of $\operatorname{ord}(p, m)$ change as m ranges over the set of π-numbers?

First we consider the case where π consists of a single prime q. A proof of the following result under the assumption that q is odd may be found in Le Veque [22], p. 52, Theorem 4-6. The extension to the case $q = 2$ presents little difficulty, so we omit the proof.

In the statement of the lemma we use the notation

$$a^n \| b$$

for positive integers a, b, n to mean that a^n divides b (denoted by $a^n | b$) but a^{n+1} does not divide b.

4.25 LEMMA. *Let p and q be distinct primes and write*

$$e(n) = \operatorname{ord}(p, q^n)$$

for $n = 1, 2, 3, \ldots$. *Define the positive integer* d *by*

$$q^d \| p^{e(1)}-1 \quad \text{if} \quad q \neq 2 \quad \text{or if} \quad p \equiv 1 \pmod 4 ,$$

$$2^d \| p^2-1 \quad \text{if} \quad q = 2 \quad \text{and} \quad p \equiv 3 \pmod 4 .$$

Then

(i) $e(n+1) = q.e(n)$ *for all* $n \geq d$,

(ii) $e(1) \leq e(2) = e(3) = \ldots = e(d)$, *with equality except in the case where* $q = 2$ *and* $p \equiv 3 \pmod 4$. □

To obtain the analogue of Lemma 4.25 when π is an arbitrary finite set of primes we make use of the following lemma.

4.26 LEMMA. *Let* $p, q_1, \ldots, q_r$ *be distinct primes and suppose*

$$m = q_1^{\alpha(1)} \ldots q_r^{\alpha(r)} ,$$

for positive integers $\alpha(1), \ldots, \alpha(r)$. *Then* $\mathrm{ord}(p, m)$ *is the least common multiple of the numbers* $\mathrm{ord}\left(p, q_i^{\alpha(i)}\right)$, *for* $i = 1, 2, \ldots, r$.

Proof. Let us write

$$k = \mathrm{ord}(p, m) ,$$

$$k(i) = \mathrm{ord}\left(p, q_i^{\alpha(i)}\right) ,$$

$$l = [k(1), \ldots, k(r)] ,$$

where $[k(1), \ldots, k(r)]$ denotes the l.c.m. of $k(1), \ldots, k(r)$.

Then

$$p^k \equiv 1 \pmod m$$

and so

$$p^k \equiv 1 \left(\mathrm{mod}\ q_i^{\alpha(i)}\right) \quad \text{for} \quad 1 \leq i \leq r .$$

From the definition of $k(i)$ it follows that

$$k(i) | k$$

for each i , and we deduce that

$$l \mid k \, .$$

Also since $k(i) \mid l$ for each i we have

$$p^{k(i)}-1 \mid p^{l}-1$$

and hence

$$p^{l} \equiv 1 \pmod{m}$$

giving

$$k \mid l \, .$$

Thus $k = l$, and the proof is complete. □

We shall also need the following property of the least common multiple.

4.27 LEMMA. *Let* $q_1, \ldots, q_r$ *be distinct primes and let* $a_1, \ldots, a_r$ *be positive integers. There is a positive integer*

$$u = q_1^{k(1)} \ldots q_r^{k(r)}$$

such that if $\alpha(1), \ldots, \alpha(r)$ *are positive integers with*

$$\alpha(i) \geq k(i)$$

for each i, *then*

$$\left[q_1^{\alpha(1)}a_1, \ldots, q_r^{\alpha(r)}a_r\right] = (1/u)q_1^{\alpha(1)} \ldots q_r^{\alpha(r)}\left[a_1, \ldots, a_r\right] .$$

Proof. We may clearly assume that $a_1, \ldots, a_r$ are not divisible by primes other than $q_1, \ldots, q_r$. Suppose that, for $1 \leq i \leq r$, the number a_i has prime factorization

$$a_i = q_1^{\beta(i,1)} \ldots q_r^{\beta(i,r)} ,$$

where $\beta(i, j) \geq 0$, for each i and j. Then the l.c.m. of $a_1, \ldots, a_r$ is given by

$$\left[a_1, \ldots, a_r\right] = q_1^{\beta(1)} \ldots q_r^{\beta(r)}$$

where

$$\beta(j) = \max\{\beta(i, j) : 1 \le i \le r\} .$$

Define non-negative integers $k(1), \ldots, k(r)$ by

$$k(i) = \beta(i) - \beta(i, i)$$

for $1 \le i \le r$, and set

$$u = q_1^{k(1)} \ldots q_r^{k(r)} .$$

If the numbers $\alpha(1), \ldots, \alpha(r)$ satisfy

$$\alpha(i) \ge k(i)$$

for $1 \le i \le r$, then

$$\alpha(i) + \beta(i, i) \ge \beta(i) ,$$

so that the l.c.m. of the numbers $q_1^{\alpha(1)}a_1, \ldots, q_r^{\alpha(r)}a_r$ is given by

$$\left[q_1^{\alpha(1)}a_1, \ldots, q_r^{\alpha(r)}a_r\right] = \prod_i q_i^{\alpha(i)+\beta(i,i)}$$

$$= \prod_i q_i^{\alpha(i)-k(i)+\beta(i)}$$

$$= (1/u)q_1^{\alpha(1)} \ldots q_r^{\alpha(r)}\left[a_1, \ldots, a_r\right]$$

which is the result required. □

Combining these last three lemmas, we can now prove the following result, which underlies much of the work in the rest of this chapter.

4.28 LEMMA. *Let* $\pi = \{q_1, \ldots, q_r\}$ *be a finite set of primes and let* p *be a prime not belonging to* π *. There is a* π*-number* m *with the property that, for any* π*-number* n *divisible by* m *, we have*

$$\operatorname{ord}(p, n) = (n/m)\operatorname{ord}(p, m) .$$

Proof. By Lemma 4.25 there are positive integers $d(1), \ldots, d(r)$ such that whenever $n \ge d(i)$ we have

$$\operatorname{ord}\left(p, q_i^{n+1}\right) = q.\operatorname{ord}\left(p, q_i^n\right) .$$

Write

$$a_i = \operatorname{ord}\left(p,\ q_i^{d(i)}\right) \text{, for } 1 \le i \le r \text{ .}$$

If $t \ge d(i)$ then we have

$$\operatorname{ord}\left(p,\ q_i^{t}\right) = q_i^{t-d(i)} a_i \text{ .} \tag{1}$$

Now by Lemma 4.27 there are non-negative integers $k(1), \ldots, k(r)$ with the property that, for any integers $\alpha(1), \ldots, \alpha(r)$ satisfying

$$\alpha(i) \ge k(i) \text{ , for } 1 \le i \le r \text{ ,}$$

we have

$$\left[q_1^{\alpha(1)} a_1, \ldots, q_r^{\alpha(r)} a_r\right] = (1/u) q_1^{\alpha(1)} \ldots q_r^{\alpha(r)} \left[a_1, \ldots, a_r\right] \text{ ,} \tag{2}$$

where u is independent of $\alpha(1), \ldots, \alpha(r)$.

Let $m = q_1^{d(1)+k(1)} \ldots q_r^{d(r)+k(r)}$. If n is a π-number divisible by m then n has the form

$$n = q_1^{s(1)} \ldots q_r^{s(r)} \text{ ,}$$

where $s(i) \ge d(i) + k(i)$ for each i.

Using Lemma 4.26 we find that

$$\begin{aligned} \operatorname{ord}(p,\ n) &= \left[\operatorname{ord}\left(p,\ q_1^{s(1)}\right), \ldots, \operatorname{ord}\left(p,\ q_r^{s(r)}\right)\right] \\ &= \left[q_1^{s(1)-d(1)} a_1, \ldots, q_r^{s(r)-d(r)} a_r\right] \quad \text{by (1)} \\ &= (1/u) q_1^{s(1)-d(1)} \ldots q_r^{s(r)-d(r)} \left[a_1, \ldots, a_r\right] \end{aligned}$$

by equation (2). Also

$$\operatorname{ord}(p,\ m) = (1/u) q_1^{k(1)} \ldots q_r^{k(r)} \left[a_1, \ldots, a_r\right]$$

by the same argument.

Hence

$$\begin{aligned} \operatorname{ord}(p,\ n)/\operatorname{ord}(p,\ m) &= \prod_i q_i^{s(i)-d(i)-k(i)} \\ &= n/m \text{ ,} \end{aligned}$$

giving the required result. □

With all these lemmas at our disposal, we can now turn to the proof of the first theorem of this chapter, which concerns systems of imprimitivity of irreducible modules for radicable abelian groups satisfying Min.

4.29 THEOREM. *Let A be a radicable abelian group satisfying* Min *and let V be a non-trivial irreducible Z_pA-module, for some prime p. Let π be the set of all primes q such that $A/C_A(V)$ has an element of order q.*

There is a π-number m, depending only on p and π, with the following property: if F is any finite subgroup of A such that $|F/C_F(V)|$ is divisible by m, then V has a system of imprimitivity consisting of conjugates of an irreducible Z_pF-module U, and

$$N_A(U) = F.C_A(U) = F.C_A(V) .$$

Proof. Assume first that $C_A(V) = 1$, i.e. that A acts faithfully on V. Then by Lemma 4.22 there is a monomorphism θ from A into the multiplicative group of the field $GF(p^\infty)$ such that V is isomorphic to the Z_pA-module $K(\theta)$ defined in 4.21. Without loss of generality we may therefore suppose that $V = K(\theta)$.

For any subgroup A_1 of A the irreducible Z_pA_1-module $K(\theta_1)$ associated with the restriction θ_1 of θ to A_1 is a subspace of $K(\theta)$. Moreover if A_2 is a second subgroup of A with $A_1 \leq A_2$, and if θ_2 denotes the restriction of θ to A_2, then we have

$$K(\theta_1) \subseteq K(\theta_2) .$$

Thus the mapping $A_1 \mapsto K(\theta_1)$ is an order-preserving correspondence between the subgroups of A and certain subspaces of $K(\theta)$.

Now our assumption that A acts faithfully on $V = K(\theta)$ implies that A is a locally cyclic p'-group, by Lemma 4.22 *(iii)*. Since A is also radicable, we have

$$A = \underset{q \in \pi}{\mathrm{Dr}}\ A_q ,$$

where, for each $q \in \pi$, A_q is a group of type C_{q^∞}. Moreover π is a finite set since A satisfies Min.

Using Lemma 4.28 we now choose a π-number m such that, for every π-number n divisible by m, we have

$$\mathrm{ord}(p, n) = (n/m)\mathrm{ord}(p, m) .$$

Suppose F is a finite subgroup of A having order k_0 divisible by m. We choose finite subgroups $F_0, F_1, F_2, \ldots$ such that

$$F = F_0 \leq F_1 \leq F_2 \leq \ldots$$

and

$$A = \bigcup_{i \geq 0} F_i ,$$

and we let

$$K(\theta_0) \subseteq K(\theta_1) \subseteq K(\theta_2) \subseteq \ldots$$

be the chain of associated subspaces of $K(\theta)$, with θ_i denoting the restriction of θ to F_i for each i. Every $K(\theta_i)$ is an irreducible Z_pF_i-module, and F_i acts faithfully on $K(\theta_i)$ for each i. Since $0 \neq K(\theta_0) \subseteq K(\theta_i)$, we have

$$K(\theta_i) = K(\theta_0).F_i \quad (i = 1, 2, 3, \ldots)$$

and similarly

$$K(\theta) = K(\theta_0).A .$$

Let us write $k_i = |F_i|$, for $i = 0, 1, 2, \ldots$. By Lemma

4.22 *(iv)* we have

$$\dim K(\theta_i) = \operatorname{ord}(p, k_i) .$$

Since $F_0, F_1, F_2, \ldots$ are all subgroups of A, the numbers $k_0, k_1, k_2, \ldots$ are all π-numbers; and each one is divisible by m, for by hypothesis m divides $k_0 = |F|$. Thus by our choice of m we have

$$\begin{aligned} \dim K(\theta_i) &= \operatorname{ord}(p, k_i) \\ &= (k_i/m)\operatorname{ord}(p, m) \end{aligned}$$

for each i, and hence

$$\begin{aligned} \dim K(\theta_i) &= (k_i/k_0) \dim K(\theta_0) \\ &= |F_i : F_0| \dim K(\theta_0) . \end{aligned}$$

We are thus in a position to apply Lemma 4.24. Setting $U = K(\theta_0)$, we deduce that V has a system of imprimitivity consisting of conjugates of the irreducible Z_pF-module U, and that $\mathrm{N}_A(U) = F$, as required. This completes the proof of the case where $\mathrm{C}_A(V) = 1$.

Suppose now that $\mathrm{C}_A(V) \neq 1$. Applying the above argument to the factor-group $A/\mathrm{C}_A(V)$, we can find a π-number m such that the conclusions of the theorem are valid with A replaced by $A/\mathrm{C}_A(V)$. Suppose that F is a finite subgroup of A such that $|F/\mathrm{C}_F(V)|$ is divisible by m, and set $F^* = F.\mathrm{C}_A(V)$. Then

$$F/\mathrm{C}_F(V) = F/(F \cap \mathrm{C}_A(V)) \cong F^*/\mathrm{C}_A(V)$$

so that m divides $|F^*/\mathrm{C}_A(V)|$, and hence V has a system of imprimitivity consisting of conjugates of some irreducible

Z_pF^*-module U , and

$$N_A(U) = F^* = F.C_A(V) .$$

As V is a direct sum of conjugates of U we have $C_A(U) = C_A(V)$, and so also

$$N_A(U) = F.C_A(U) .$$

Since U is clearly also irreducible as a Z_pF-module we thus obtain the required result. □

4.3 Systems of Imprimitivity for Indecomposable Modules

In section 4.2 we studied systems of imprimitivity for irreducible Z_pA-modules, where A was a radicable abelian group satisfying Min . We now study the systems of imprimitivity for the indecomposable modules which are associated with quasi-radicable metabelian groups satisfying Min-n . We prove a generalization of Theorem 4.29 which is valid for these indecomposable modules: this will constitute the main step in the proof of our ultimate goal, Theorem 4.41.

We first quote the following result, which is essentially Lemma 2.3 of Hartley and McDougall [12], and plays an important role in the results of this section.

4.31 LEMMA. *Let* p *be a prime and* A *an abelian* p'*-group, and let* W *be an irreducible* Z_pA*-module.*

(i) The divisible hull $\bar{W}$ *of* W *admits an* A*-module structure extending that on* W *such that the only proper submodules of* $\bar{W}$ *are the submodules* $\bar{W}[p^n]$ *, for* $n = 0, 1, 2, \ldots$.

(ii) If V *is any indecomposable* A*-module having a submodule isomorphic to* W *, then any monomorphism from this submodule into* $\bar{W}$

can be extended to a monomorphism from V *into* $\bar{W}$. □

Notice that *(ii)* implies that two different A-module structures on $\bar{W}$ extending that on W must give rise to isomorphic modules.

The next lemma provides the key to the proof of Theorem 4.33.

4.32 LEMMA. *Let* A *be a group and* B *a subgroup, and let* V *be an* A*-module having a system of imprimitivity consisting of conjugates of some* B*-module* U . *Suppose that*

$$\mathrm{N}_A(U) = B.\mathrm{C}_A(U)$$

and suppose further that the divisible hull $\bar{U}$ *of* U *admits a* B*-module structure extending that on* U , *with the property that*

$$\mathrm{C}_B(\bar{U}) = \mathrm{C}_B(U) .$$

Then the divisible hull $\bar{V}$ *of* V *admits an* A*-module structure extending that on* V , *and the conjugates of* $\bar{U}$ *form a system of imprimitivity for* $\bar{V}$. *Moreover*

$$\mathrm{N}_A(\bar{U}) = \mathrm{N}_A(U)$$

and

$$\mathrm{C}_A(\bar{U}) = \mathrm{C}_A(U) .$$

If in addition the underlying group of V *has prime exponent* p , *then for each positive integer* n *the conjugates of* $\bar{U}[p^n]$ *form a system of imprimitivity for* $\bar{V}[p^n]$ *and*

$$\mathrm{N}_A(\bar{U}[p^n]) = \mathrm{N}_A(U)$$

and

$$\mathrm{C}_A(\bar{U}[p^n]) = \mathrm{C}_A(U) .$$

Proof. Let $N = \mathrm{N}_A(U)$. Since $N = B.\mathrm{C}_A(U)$, we can view U as an N-module. By assumption $\bar{U}$ has a B-module structure extending that on U , and we can give $\bar{U}$ an N-module structure by making

$C_A(U)$ act trivially also on $\bar{U}$. This gives a well-defined action in view of our assumption that

$$C_B(\bar{U}) = C_B(U) = B \cap C_A(U) .$$

Now let T be a transversal to N in A. The set

$$\{Ut : t \in T\}$$

is a complete set of conjugates of U, and so

$$V = \bigoplus_{t \in T} Ut . \tag{1}$$

Let $V' = U \otimes_{\mathbb{Z}N} \mathbb{Z}A$ be the induced A-module associated with U; then V' has a direct decomposition as an abelian group

$$V' = \bigoplus_{t \in T} U \otimes t \tag{2}$$

and we obtain an A-isomorphism between V' and V by extending the mapping

$$u \otimes t \longmapsto ut \quad (u \in U,\ t \in T)$$

to all of V' by linearity.

Now V' is a submodule of the induced module

$$\bar{V}' = \bar{U} \otimes_{\mathbb{Z}N} \mathbb{Z}A$$

and as above we have an abelian group direct decomposition

$$\bar{V}' = \bigoplus_{t \in T} \bar{U} \otimes t . \tag{3}$$

But $\bar{U} \otimes t$ is evidently the divisible hull of $U \otimes t$, for each $t \in T$. Since the divisible hull of a direct sum of abelian groups is the direct sum of the divisible hulls of the factors, we have an isomorphism of abelian groups

$$\bar{V} \cong \bar{V}' ,$$

extending the A-isomorphism $V \cong V'$ described above. Therefore we can use the A-module structure on $\bar{V}'$ to endow $\bar{V}$ with an A-module structure extending that on V. We then obtain an equation

$$\bar{V} = \bigoplus_{t \in T} \bar{U}t \tag{4}$$

corresponding to (1). Hence the conjugates of $\bar{U}$ form a system of imprimitivity for $\bar{V}$ and we have

$$N_A(\bar{U}) = N_A(U) = N \ . \tag{5}$$

By our definition of the N-module structure on $\bar{U}$ we have

$$C_N(\bar{U}) = C_A(U)$$

and from (5) we have $C_A(\bar{U}) \le N$ so we deduce that

$$C_A(\bar{U}) = C_A(U) \ , \tag{6}$$

as required.

Finally suppose that the underlying group of V has exponent p. From (4) and (5) we obtain

$$\bar{V}[p^n] = \bigoplus_{t \in T} \bar{U}[p^n]t$$

and

$$N_A(\bar{U}[p^n]) = N_A(U) = N \ .$$

Also we have

$$U = U[p] \le \bar{U}[p^n] \le \bar{U}$$

so from (6) we deduce that

$$C_A(\bar{U}[p^n]) = C_A(U) \ ,$$

completing the proof. □

4.33 THEOREM. *Let* A *be a radicable abelian group satisfying* Min *and let* V *be a non-trivial indecomposable* A*-module satisfying* Min-A. *Let* π *be the set of primes* q *such that* $A/C_A(V)$ *has an element of order* q *and suppose that the underlying group of* V *is a* p*-group for some prime* p *not belonging to* π.

There is a π*-number* m, *depending only on* p *and* π, *with* *the following property: if F is any finite subgroup of A such that $|F/C_F(V)|$ is divisible by m, then there is an indecomposable F-module U satisfying* Min-*F such that*

$$N_A(U) = F.C_A(U) = F.C_A(V)$$

and the conjugates of U in A form a system of imprimitivity for V.

Proof. We assume that $C_A(V) = 1$: the general case follows from this case exactly as in the proof of Theorem 4.29.

Since V satisfies Min-A it has an irreducible submodule W. Clearly $pW = 0$, so we may view W also as a $\mathbb{Z}_pA$-module. Lemma 4.31 now shows that the divisible hull $\bar{W}$ of W admits an A-module structure extending that on W, and that V is isomorphic to a submodule of $\bar{W}$. Since the only proper submodules of $\bar{W}$ are the submodules $\bar{W}[p^n]$, for $n = 0, 1, 2, \ldots$, it will be enough for the proof of the theorem to show that $\bar{W}$ can be given an A-module structure extending that on W such that the conclusions of the theorem are valid with V replaced by $\bar{W}$ or by any of the submodules $\bar{W}[p^n]$, $n = 0, 1, 2, \ldots$.

Now the module W satisfies the conditions of Theorem 4.29: let m be the π-number associated with W by Theorem 4.29. Suppose F is a finite subgroup of A whose order is divisible by m. Theorem 4.29 shows that there is an irreducible F-module U with

$$N_A(U) = F.C_A(U) = F$$

such that the conjugates of U in A form a system of imprimitivity for W. Clearly $C_F(U) = 1$.

By Lemma 4.31 the divisible hull $\bar{U}$ of U admits an F-module structure extending that on U such that

$$C_F(\bar{U}) = C_F(U) = 1$$

and we also conclude from Lemma 4.31 that the only proper F-submodules of $\bar{U}$ are the submodules $\bar{U}[p^n]$, for $n = 0, 1, 2, \ldots$. In particular $\bar{U}$ is an indecomposable F-module satisfying Min-F and the same is true of each of the submodules $\bar{U}[p^n]$.

Now we are in a position to apply Lemma 4.32. We conclude from this that $\bar{W}$ has an A-module structure extending that on W such that the conjugates of $\bar{U}$ form a system of imprimitivity for $\bar{W}$ and

$$N_A(\bar{U}) = N_A(U) = F \ ;$$

also, for each $n \geq 1$, the conjugates of $\bar{U}[p^n]$ form a system of imprimitivity for $\bar{W}[p^n]$ and

$$N_A(\bar{U}[p^n]) = N_A(U) = F \ .$$

The A-module structure we have assigned to $\bar{W}$ depended on the choice of the finite subgroup F . However, as we have already noted, different A-module structures on $\bar{W}$ extending that on W give rise to isomorphic modules. Since the properties we wish to establish are isomorphism-invariant, it follows that the conclusions of the theorem are valid for $\bar{W}$ and its submodules $\bar{W}[p^n]$ $(n = 0, 1, 2, \ldots)$, independently of the choice of A-module structure. Therefore the theorem is proved. □

4.4 The Main Theorem

We now apply the results of sections 4.2 and 4.3 to deduce the main theorem of this chapter. For convenience we repeat the statement of the theorem again here.

4.41 THEOREM. *Let* G *be a quasi-radicable metabelian group* *satisfying* Min-n *and let* A *be a complement to* G' *in* G . *Then* G' *is expressible as a direct product*

$$G' = V_1 \times \dots \times V_n$$

of a finite number of normal subgroups $V_1, \dots, V_n$ *of* G *such that, for* $i = 1, 2, \dots, n$, *the group*

$$G_i = V_i A$$

has the following structure:

$$G_i = \left(U_i \operatorname{wr}_{F_i} A_i\right) \times B_i$$

where U_i, F_i, A_i *and* B_i *are subgroups such that*

(i) $A = A_i \times B_i$,

(ii) F_i *is finite,*

(iii) $U_i F_i \times B_i$ *is a group satisfying* Min .

Proof. If G is abelian the result is trivial, so suppose $G' \neq 1$. Write $V = G'$. We may view V as an A-module, with the elements of A acting on V by conjugation. Since G satisfies Min-n , V satisfies Min-A and hence is a direct product

$$V = V_1 \times \dots \times V_n$$

of a finite number of indecomposable A-submodules $V_1, \dots, V_n$.

To prove that $V_1, \dots, V_n$ have the properties described in the theorem it is clearly enough to deal with the case $n = 1$. Therefore we now assume that V itself is an indecomposable A-module.

By Baer's Theorem (Theorem 3.15) G is periodic, and it follows that the underlying group of V must be a p-group for some prime p . Theorem 3.31 shows that each Sylow p-subgroup of G is abelian, so $A/C_A(V)$ is a p'-group. Also $C_A(V) < A$ because we have assumed

G non-abelian.

Now $C_A(V)$ is an abelian group satisfying Min so we can write

$$C_A(V) = B_1 \times C_1$$

where B_1 is radicable and C_1 is finite. Since a radicable subgroup of an abelian group is a direct summand, B_1 also has a direct complement A_1 in A, and we may further assume that $C_1 \leq A_1$. As $[B_1, V] = 1$ the subgroup B_1 is normal in G and we have

$$G = VA_1 \times B_1 . \tag{1}$$

We may now view V also as an indecomposable A_1-module. We have $C_{A_1}(V) = A_1 \cap C_A(V) = C_1$, so $C_{A_1}(V)$ is finite. Furthermore, $A_1/C_{A_1}(V)$ is isomorphic to $A/C_A(V)$ and so is also a non-trivial radicable p'-group.

The conditions of Theorem 4.33 are now satisfied by V and A_1. Suppose π is the set of primes q such that $A_1/C_{A_1}(V)$ has an element of order q, and let m be the π-number associated with V and A_1 by Theorem 4.33. The group $A_1/C_{A_1}(V)$ has a direct summand of type C_{q^∞} for each prime q in π, and hence has also a finite subgroup $F/C_{A_1}(V)$ of order divisible by m. Since $C_{A_1}(V)$ is finite, the subgroup F is also finite. Theorem 4.33 shows that there is an indecomposable F-module U satisfying Min-F such that the conjugates of U form a system of imprimitivity for V and

$$N_A(U) = F.C_A(V) = F .$$

Since the systems of imprimitivity for V as a module coincide with its systems of imprimitivity as a normal subgroup, we see that the group VA_1 satisfies the criterion of Lemma 4.13 for a group to be expressible as a twisted wreath product. Thus we have

$$VA_1 = U \operatorname{wr}_F A_1 ,$$

so that by (1), G is expressible in the form

$$G = \left(U \operatorname{wr}_F A_1\right) \times B_1 .$$

We already know that $A = A_1 \times B_1$ and that F is finite, so to prove the theorem it remains to establish that UF is a group satisfying Min .

Now U satisfies Min-F by its construction, and F is finite, so the group UF satisfies Min-n . As $|UF : U| = |F|$ is finite, Wilson's theorem (Theorem 3.21) shows that U also satisfies Min-n . But U is abelian, so U satisfies Min and therefore so does UF . This completes the proof of Theorem 4.41. □

With certain extra restrictions on the groups involved, Theorem 4.41 takes on a simpler form. In particular, the following result is immediate from Theorem 4.41.

4.42 THEOREM. *Let G be a quasi-radicable metabelian group satisfying* Min-n *and suppose that*

(i) G is directly indecomposable,

(ii) G' is directly indecomposable as a G-module (i.e. G' is not a direct product of two non-trivial normal subgroups of G).

Then G is expressible as a twisted wreath product

$$G = U \operatorname{wr}_F A ,$$

where A is a radicable abelian group satisfying Min *and UF is* *also a group satisfying* Min , *with* F *a finite subgroup of* A *and* U *a subgroup of* G' . □

We note in particular that the conditions *(i)* and *(ii)* of Theorem 4.42 are satisfied when G is *monolithic* (i.e. when the intersection of the non-trivial normal subgroups of G is non-trivial). This shows that the description of Theorem 4.42 applies to Čarin's groups (see section 3.61) for these groups are certainly monolithic.

CHAPTER FIVE

A METHOD FOR CONSTRUCTING GROUPS SATISFYING MIN-$\mathcal{N}$

We describe in this chapter a method for constructing groups satisfying Min-n by means of a process of embedding wreath products into treble products. Using this method we obtain examples of quasi-radicable soluble groups of derived length three which satisfy Min-n but have Sylow p-subgroups which are not hypercentral. Also we construct quasi-radicable soluble groups of arbitrarily large derived lengths each of whose normal subgroups form a well-ordered chain (when ordered by set-theoretic inclusion) consisting of the terms of the derived series together with subgroups containing the derived group. In each of these groups the derived series is also the shortest series with nilpotent factors: hence there is also no bound to the nilpotent lengths of the groups.

By the same process we are able to establish the existence of $2^{\aleph_0}$ pairwise non-isomorphic periodic locally soluble perfect groups in each of which the normal subgroups form a chain of order type $\omega + 1$. An example due to McLain [26] establishes the existence of two non-isomorphic groups with these properties, for the group constructed by McLain has precisely two isomorphism classes of non-trivial homomorphic images. We show that, in contrast to this, the class of groups we construct includes an uncountable number of Hopfian groups.

5.1 The 3-step Soluble Case

We recall from Chapter 2 that any treble product

$$T = \mathrm{Tr}(A, B, C; \sigma, \tau)$$

of three groups A, B, C has the property that $\langle A, C\rangle$ is a wreath product $A \text{ wr } C$. The groups we construct in this section are obtained by starting from a wreath product $W = A \text{ wr } C$, and defining homomorphisms

$$\sigma : B \to \mathrm{Aut}\, A ,$$
$$\tau : C \to \mathrm{Aut}\, B ,$$

for some group B, such that W is embedded in the treble product

$$T = \mathrm{Tr}(A, B, C; \sigma, \tau) .$$

In particular we shall show that if A is a cyclic group of prime order then it is often possible to carry out this embedding so that the base group of W becomes a minimal normal subgroup of T.

To do this we make use of the following lemma, which is of great importance for this chapter.

5.11 LEMMA. *Let* $G = \mathrm{Tr}(A, B, C; \sigma, \tau)$ *, where* A *is a cyclic minimal normal subgroup of* AB *. If the condition*

$$\mathrm{C}_C\{B/\mathrm{C}_B(A)\} = 1$$

is satisfied in G *, then* A^G *is a minimal normal subgroup of* G *.*

Proof. Write $C^* = \mathrm{C}_C\{B/\mathrm{C}_B(A)\}$, and suppose there is a normal subgroup N of G with $1 \neq N < A^G$. We shall deduce from this that $C^* \neq 1$.

Since $A^G = \mathrm{Dr}_{c \in C}\, A^c$, each non-trivial element $x \in A^G$ can be expressed as a product

$$x = a_1^{c_1} a_2^{c_2} \cdots a_n^{c_n} \tag{1}$$

where $a_1, \ldots, a_n$ are non-trivial elements of A and $c_1, \ldots, c_n$ are distinct elements of C , and this expression is unique apart from the ordering of the terms $a_1^{c_1}, \ldots, a_n^{c_n}$. We shall refer to the integer n as the *span* of x and denote it by $s(x)$.

Choose an element $x \neq 1$ in N of minimal span, say $s(x) = n$, and suppose (1) is the expression for x in terms of its projections in the factors A^c. Since any conjugate of x in C has the same span and also lies in N, we may suppose that $c_1 = 1$. If $n = 1$, then $x \in A$ and hence

$$N \geq \langle x \rangle^G = A^G ,$$

because A is a minimal normal subgroup of AB. But this contradicts our choice of N. Therefore $n > 1$.

We claim that $c_n \in C^*$. To prove this, we suppose on the contrary that $c_n \notin C^*$. Then also $c_n^{-1} \notin C^*$, so there is an element $b \in B$ such that

$$\left[b, c_n^{-1}\right] \notin \mathrm{C}_B(A) .$$

As $\mathrm{C}_B(A) = \ker \sigma$, this is equivalent to

$$b^\sigma \neq b^{c_n^{-1}\sigma} . \tag{2}$$

Now A is a cyclic group, so there is an integer $m \neq 0$ such that

$$a^b = a^{b^\sigma} = a^m \tag{3}$$

for each $a \in A$. Let $y = x^{-m}x^b$; then $y \in N$ and we have

$$y = \left(a_1^m a_2^{mc_2} \ldots a_n^{mc_n}\right)^{-1} \left(a_1^b a_2^{c_2 b} \ldots a_n^{c_n b}\right)$$

$$= \left(a_1^{-m} a_1^b\right) \left(a_2^{-mc_2} a_2^{b_2 c_2}\right) \ldots \left(a_n^{-mc_n} a_n^{b_n c_n}\right) ,$$

where $b_i = b^{c_i^{-1}}$ for $2 \le i \le n$. Thus the projection of y in the factor A of the direct product $\mathrm{Dr}_{c \in C} A^c$ is

$$a_1^{-m} a_1^{b} = \left(a_1^{b^\sigma}\right)^{-1} a_1^{b} = 1$$

by (3). On the other hand, the projection of y in A^{c_n} is

$$a_n^{-m} a_n^{b_n} = \left(a_n^{b^\sigma}\right)^{-1} \left(a_n^{b_n^\sigma}\right) \ne 1$$

since b^σ and $b_n^\sigma = b^{c_n^{-1}\sigma}$ are distinct automorphisms of A by (2), and A is generated by the element a_n. Therefore y is a non-trivial element of N with span less than n, and we have a contradiction to our choice of x. We conclude that $1 \ne c_n \in C^*$, and from this the required result follows immediately. □

Unfortunately Lemma 5.11 no longer remains true if the word 'cyclic' is omitted, as the following example shows.

EXAMPLE. Let $G = \mathrm{Tr}(A, B, C; \sigma, \tau)$, where $A = \langle a_1 \rangle \times \langle a_2 \rangle$ is a direct product of two 2-cycles, $B = \langle b \rangle$ has order 3 and $C = \langle c \rangle$ has order 2, and the homomorphisms σ, τ are such that the relations

$$a_1^b = a_2, \quad a_2^b = a_1 a_2,$$

$$b^c = b^{-1}$$

are valid in G. Then the split extensions AB and BC are isomorphic to the alternating group A_4 and the symmetric group S_3

respectively. Hence A is a minimal normal subgroup of AB and $C_B(A) = C_C(B) = 1$, so that we certainly have

$$C_C(B/C_B(A)) = 1 .$$

However, $A^G = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_1^c \rangle \times \langle a_2^c \rangle$ is not a minimal normal subgroup of G since it properly contains the non-trivial normal subgroup $\langle a_1 a_2^c \rangle \times \langle a_1^c a_2 \rangle$.

In applying Lemma 5.11 we shall take C to be a locally cyclic q-group, for some prime q, and show that we can choose an elementary abelian group for B so that the normal subgroups of G form a well-ordered chain. To do this we need to ensure that the normal subgroups of the split extension BC form a well-ordered chain, since BC is a homomorphic image of G. We now consider a class of metabelian groups which furnishes some candidates for the group BC.

The following term was introduced by B.H. Neumann [27].

5.12 DEFINITION. A group G is said to be a *just metabelian group* if G is metabelian and every proper homomorphic image of G is abelian, but G itself is non-abelian.

The particular just metabelian groups we consider belong to the class of just metabelian groups with trivial centre, studied in detail by M.F. Newman in [31]. Following Newman we refer to these latter groups as *JM-groups*. We now quote without proof some facts concerning *JM*-groups established in [31], which it will be convenient to have at our disposal.

5.13 LEMMA ([31], Theorems 3.4 and 3.5). *Let G be a JM-group. Then*

(i) G splits over its derived group G',

(ii) G' coincides with its centralizer in G, and

$$G' = [G', c]$$

for each element $c \in C$ *not belonging to* G' . $\square$

The groups we now define are, to within isomorphism, precisely the groups of linear inhomogeneous substitutions studied in section 5 of [31].

5.14 DEFINITION. Let Ω be a field and Λ a subgroup of the multiplicative group of Ω . Denote by $L(\Lambda, \Omega)$ the subgroup of GL(2, Ω) consisting of all matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ \omega & 1 \end{pmatrix}$$

where $\lambda \in \Lambda$ and $\omega \in \Omega$.

The following result is immediately deducible from Theorem 5.3 of [31]: we omit the proof.

5.15 THEOREM. *If* $L(\Lambda, \Omega)$ *is non-abelian and the elements of* Λ *generate the additive group of* Ω *, then* $L(\Lambda, \Omega)$ *is a JM-group. In this case the derived group of* $L(\Lambda, \Omega)$ *is isomorphic to the additive group of* Ω *and is complemented by a subgroup isomorphic to* Λ . $\square$

We now define a special class of groups of the type considered in Theorem 5.15.

5.16 DEFINITION. Let p and q be distinct primes and let n be a positive integer or the symbol ∞ . Let F_n be the field obtained from the prime field $F_0 = \mathrm{GF}(p)$ by adjoining a primitive q^k-th root of unity for each positive integer k not exceeding n . (When $n = \infty$ we mean by this that we adjoin a primitive q^k-th root of unity for every positive integer k .) Let Λ_n be the subgroup of the multiplicative group of F_n generated by these primitive roots of unity. We write $H(p, q^n)$ to denote the group $L(\Lambda_n, F_n)$ associated with Λ_n and F_n.

We can now easily establish the following properties of the groups $H(p, q^n)$.

5.17 LEMMA. *Let p and q be distinct primes and let $H = H(p, q^n)$, where $1 \le n \le \infty$. Then H has the following properties:*

(i) H is a JM-group,

(ii) H' is an elementary abelian p-group and is complemented in H by a subgroup C of type C_{q^n},

(iii) $[H', c] = H'$ for all $c \ne 1$ in C,

(iv) the normal subgroups of H form a well-ordered chain.

Proof. It is immediate from the definition of the field F_n that the elements of Λ_n generate the additive group of F_n. Also H is non-abelian (since $|F_n| > 2$ for every value of n), so *(i)* follows from Theorem 5.15.

Theorem 5.15 also shows that H' is isomorphic to the additive group of F_n and hence must be an elementary abelian p-group, and that H' has a complement C which is isomorphic to Λ_n and hence is a group of type C_{q^n}. Thus *(ii)* is proved.

Statement *(iii)* follows from Lemma 5.13 *(ii)*.

To prove *(iv)*, suppose $1 \ne N \triangleleft H$. Because H is a JM-group, $H' \le N$, and therefore using *(ii)* we have

$$N = N \cap H'C$$
$$= H'(N \cap C) .$$

This shows that every non-trivial normal subgroup of H has the form $H'D$, where $D \le C$. As the subgroups of C form a well-ordered chain, so therefore do the normal subgroups of H. This completes the proof. $\square$

Remark. Čarin's groups are isomorphic to the groups $H(p, q^{\infty})$, as we see from the representation by matrices described in Chapter 3 (section 3.61).

We now make use of the properties of the groups $H(p, q^{n})$ to prove the main theorem of this section.

5.18 THEOREM. *Let q_0 and q_1 be (not necessarily distinct) primes and n a positive integer or the symbol ∞, and let A be a cyclic group of order q_0 and C a group of type $C_{q_1^n}$.*

If p_1 is any prime such that $p_1 | q_0 - 1$ and $p_1 \ne q_1$, then the wreath product

$$W = A \operatorname{wr} C$$

can be embedded in a treble product

$$G = \operatorname{Tr}(A, B, C; \sigma, \tau)$$

in which B is an elementary abelian p_1-group, so that G has the following properties:

(a) $G'' = A^C$ *and* $G' = A^C B$,

(b) the normal subgroups of G form a well-ordered chain consisting of the terms of the derived series and subgroups of the form $G'D$, where $D \le C$,

(c) $C_G\left(G^{(i)}/G^{(i+1)}\right) = G^{(i)}$ *for* $i = 0, 1, 2$.

Proof. Let $H = H\left(p_1, q_1^n\right)$. By Lemma 5.17 *(ii)*, H' is an elementary abelian p_1-group and is complemented in H by a group isomorphic to C. We take B to be a group isomorphic to H' and define a homomorphism

$$\tau : C \to \operatorname{Aut} B$$

such that the split extension $B \underset{\tau}{]} C$ is isomorphic to H.

Since $p_1 | q_0 - 1$, the group A has an automorphism α of order p_1. Because B is an elementary abelian p_1-group, there is a homomorphism

$$\sigma : B \to \operatorname{Aut} A$$

such that $B^{\sigma} = \langle \alpha \rangle$.

With B, σ and τ defined in this way we let

$$G = \operatorname{Tr}(A, B, C; \sigma, \tau) .$$

Clearly $W \leq G$; we now show that G has the properties *(a)*, *(b)*, *(c)*.

First we show that G satisfies the conditions of Lemma 5.11. Clearly A is a cyclic minimal normal subgroup of AB. Moreover, since σ is a non-trivial homomorphism from B into $\operatorname{Aut} A$ we have

$$C_B(A) = \ker \sigma < B . \tag{1}$$

Let $C^* = C_C\left(B/C_B(A)\right)$, and suppose, if possible, that $C^* \neq 1$. Then there is an element $c \neq 1$ in C^*. Since $BC \cong H\left(p_1, q_1^n\right)$, Lemma 5.13 *(ii)* and the inequality (1) imply that

$$B = [B, c] \leq C_B(A) < B ,$$

which is obviously a contradiction. Therefore $C^* = 1$, and we infer from Lemma 5.11 that A^C is a minimal normal subgroup of G.

Now let $N \triangleleft G$ such that $A^C < N$. As $G = A^C BC$ and $A^C \cap BC = 1$, we have

$$N = A^C(N \cap BC)$$

and

$$N \cap BC \neq 1 .$$

However by Lemma 5.17 *(i)* every non-trivial normal subgroup of BC contains B, and so $B \leq N$. Thus

$$N \cap BC = B(N \cap C)$$

and consequently $N = A^C B(N \cap C)$. Hence we have shown that every normal subgroup of G which properly contains A^C has the form $A^C BD$, where $D \leq C$.

Suppose next that $1 \neq N \triangleleft G$ but $A^C \nleq N$. Then $A^C \cap N = 1$, so $N \leq \mathrm{C}_G(A^C)$. However

$$\begin{aligned} B \cap \mathrm{C}_G(A^C) &\leq B \cap \mathrm{C}_G(A) = \mathrm{C}_B(A) \\ &< B \text{ by (1)} \end{aligned}$$

and hence

$$\begin{aligned} A^C B \cap \mathrm{C}_G(A^C) &= A^C\left[B \cap \mathrm{C}_G(A^C)\right] \\ &< A^C B \end{aligned}$$

whence it follows that

$$\mathrm{C}_G(A^C) = A^C . \tag{2}$$

But this implies that $1 \neq N \leq A^C$, and hence $N = A^C$, contrary to our choice of N. This contradiction shows that every non-trivial normal subgroup of G contains A^C, and it follows from the above that the only non-trivial normal subgroups of G are A^C and subgroups of the form $A^C BD$, where $D \leq C$. As the subgroups of C form a well-ordered chain, so do the normal subgroups of G.

Now we have

$$G/A^C B \cong C ,$$

$$G/A^C \cong BC \cong H$$

and

$$A^C B/A^C \cong B ,$$

so that $G/A^C B$ is abelian but G/A^C is not, which implies that $G' = A^C B$; and similarly $A^C B/A^C$ is abelian but $A^C B$ is not, which implies that $G'' = A^C$. Combining these facts with those established in the preceding paragraph, we conclude that the only normal subgroups of G are the terms of its derived series and subgroups of the form $G'D$, where $D \leq C$. Thus we have proved *(a)* and *(b)*.

To prove *(c)* we need only consider the case $i = 1$: for $i = 0$ the assertion is trivial, and equation (2) gives the case $i = 2$. Thus we need to show that

$$C_G(G'/G'') = G' .$$

Because $A^C = G'' \leq C_G(G'/G'')$, we have

$$\begin{aligned} C_G(G'/G'') &= C_G\left(A^C B/A^C\right) = A^C BC \cap C_G\left(A^C B/A^C\right) \\ &= A^C\left[BC \cap C_G\left(A^C B/A^C\right)\right] \text{, by the modular law,} \\ &= A^C C_{BC}\left(A^C B/A^C\right) \\ &= A^C C_{BC}(B) . \end{aligned}$$

However since $BC \cong H$ and $B \cong H'$ (under the same isomorphism), Lemma 5.17 *(iii)* implies that

$$C_{BC}(B) = B ,$$

whence

$$C_G(G'/G'') = A^C B = G' ,$$

and thus *(c)* is proved. □

A special case of this theorem yields the following result.

5.19 THEOREM. *Let* A *and* C *be respectively a cyclic group of order* q_0 *and a quasicyclic* q_1*-group, where* q_0 *and* q_1 *are (not necessarily distinct) primes. If there exists a prime* p_1 *such that* $p_1 | q_0 - 1$ *and* $p_1 \neq q_1$ *then the wreath product* $A \text{ wr } C$ *can be embedded in a quasi-radicable soluble group of derived length* 3 *whose normal subgroups form a chain of order type* $\omega + 1$.

Proof. Suppose p_1 is a prime such that $p_1 | q_0 - 1$ and $p_1 \neq q_0$, and let

$$G = \mathrm{Tr}(A, B, C; \sigma, \tau)$$

be the group constructed by the method of Theorem 5.18 (taking $n = \infty$). Then the only non-trivial normal subgroups are G'' and the subgroups containing G'. But $G/G' \cong C$, and C is a quasi-cyclic group so its subgroups form a chain of order type $\omega + 1$. Therefore the normal subgroups of G also form a chain of order type $\omega + 1$.

From this it follows that the unique maximal element of this chain of normal subgroups, namely G itself, has no immediate predecessor. Hence G has no proper normal subgroups of finite index and is therefore quasi-radicable.

Finally since A, B, C are abelian, G is clearly a soluble group of derived length 3. □

COROLLARY. *Let* q *be any odd prime. There is a quasi-radicable soluble group satisfying* Min-n *which has a non-hypercentral Sylow* q*-subgroup.*

Proof. Let A be a cyclic group of order q and C a quasi-cyclic q-group. Since q is odd there is a prime p_1 such that $p_1 | q-1$ (and hence also $p_1 \neq q$). Taking $q_0 = q_1 = q$ in the theorem, we deduce that $A \text{ wr } C$ can be embedded in a quasi-radicable soluble group

$$G = \mathrm{Tr}(A, B, C; \sigma, \tau)$$

of derived length 3 which satisfies Min-n . Since B is a p_1-group, $A \text{ wr } C$ will be a Sylow q-subgroup of G . However it is well-known that a wreath product of a non-trivial group and an infinite group always has trivial centre (see e.g. P.M. Neumann [30], Corollary 3.4). Hence $A \text{ wr } C$ is a non-hypercentral Sylow q-subgroup of G . □

5.2 The General Construction

The starting-point for the construction of the rest of our examples is the following theorem, which is really only a straightforward extension of Theorem 5.18.

5.21 THEOREM. *Let* q_0 *and* q_1 *be (not necessarily distinct) primes, and let* p_1 *be a prime such that* $p_1 \neq q_1$ *and* $p_1 | q_0-1$. *Let* A *be a group such that* $|A/A'| = q_0$ *and let* C *be an abelian group of type* $C_{q_1^n}$, *where* $1 \leq n \leq \infty$.

If A *has an automorphism of order* p_1 *which acts non-trivially on* A/A' , *then the wreath product* $A \text{ wr } C$ *can be embedded in a treble product*

$$G = \mathrm{Tr}(A, B, C; \sigma, \tau)$$

in which B *is an elementary abelian* p_1*-group, so that*

$$G' = A^C B\ , \quad G'' = A^C\ , \quad G''' = (A')^C\ ,$$

and the only other normal subgroups of G *containing* G''' *are those of the form* $G'D$ *, where* $D \le C$ *.*

Proof. The construction is similar to that of Theorem 5.18.

Let $H = H\left(p_1, q_1^n\right)$. By Lemma 5.17, H' is an elementary abelian p_1-group and is complemented by a subgroup isomorphic to C . We take B to be a group isomorphic to H' and define a homomorphism

$$\tau : C \to \operatorname{Aut} B$$

such that the split extension $B \underset{\tau}{]} C$ is isomorphic to H .

By hypothesis, A has an automorphism, α say, of order p_1 which acts non-trivially on A/A' . As B is an elementary abelian p_1-group there is a homomorphism

$$\sigma : B \to \operatorname{Aut} A$$

such that $B^\sigma = \langle \alpha \rangle$. We define

$$G = \operatorname{Tr}(A, B, C; \sigma, \tau)$$

and show that G has the required properties.

Let M denote the normal closure of A' in G . Because

$$A^G = A^C = \underset{c \in C}{\operatorname{Dr}}\ A^c$$

the factor-group A^C/M is expressible as a direct product

$$A^C/M = \underset{c \in C}{\operatorname{Dr}}\ A^g (A')^c$$

and we see that

$$A^C C/M \cong (A/A')\ \mathrm{wr}\ C\ .$$

The elements of B^σ induce automorphisms on A^C/M and it can easily be verified that

$$G/M = \mathrm{Tr}(A/A', B, C; \bar{\sigma}, \tau) ,$$

where $\bar{\sigma} : B \to \mathrm{Aut}\, A/A'$ is defined by taking $b^{\bar{\sigma}}$ to be the automorphism induced on A/A' by b^{σ}, for each $b \in B$.

As A/A' is a cyclic group of order q_0 and α induces a non-trivial automorphism on A/A', the group G/M is isomorphic to one of the groups constructed by the method of Theorem 5.18. It follows from the facts proved in Theorem 5.18 that A^C/M is contained in every non-trivial normal subgroup of G/M, and G/M has derived length 3. Since G/A^C is metabelian, we have $G'' = A^C$ and hence

$$G''' = \left(A^C\right)' = (A')^C = M .$$

It now follows from Theorem 5.18 that the only normal subgroups of G properly containing G''' are G'' and subgroups of the form $G'D$, where $D \le C$. This completes the proof. □

We now investigate what more can be said about the group G of Theorem 5.21 when we impose additional restrictions on A, or on the primes q_0, q_1 and p_1. To begin with, we impose a condition on the central factors of A to ensure that the normal subgroups of G lying below G''' are easily described.

For this we need the following result, which is an analogue for treble products of a well-known fact about wreath products. (See e.g. P. Hall [9], p. 431.) The proof is exactly as for wreath products, and is very simple, so we omit it.

5.22 LEMMA. *Let* G *be a treble product*

$$G = \mathrm{Tr}(A, B, C; \sigma, \tau)$$

and let N *be a normal subgroup of* G *contained in* A^C. *Let* L *be the projection of* N *into the factor* A *of the direct product*

$A^C = \operatorname{Dr}_{c\in C} A^c$, *and let* $M = A \cap N$. *Then* L/M *is a central factor of* A. *If* $L = M$ *then* $N = M^G$. □

5.23 COROLLARY. *If the group* A *of Lemma* 5.22 *has* A/A' *as its only non-trivial central factor, then every normal subgroup of* G *contained in* $(A')^C$ *is the normal closure in* G *of its intersection with* A. □

5.24 THEOREM. *If, under the assumptions of Theorem* 5.21, *the group* A *has* A/A' *as its only non-trivial central factor, then the only normal subgroups of* G *are the subgroups containing* G' *and subgroups of the form* N^G, *where* $N \lhd A$.

Proof. By Theorem 5.21, the subgroups G'' and G''' are normal closures in G of normal subgroups of A. The group G satisfies the conditions of Corollary 5.23, so any normal subgroup of G contained in $G''' = (A')^G$ is also the normal closure in G of a normal subgroup of A. Finally, by Theorem 5.21 again, the only other normal subgroups of G are those containing G'. □

The next result will give us a method of iterating the treble product construction. It enables us to carry out the construction of the group G of Theorem 5.21 in the case $n = 1$ so that G has an automorphism of prime order which acts non-trivially on G/G'. Then we may take G in place of A and use Theorem 5.21 to construct a second group with similar properties but of larger derived length, and so on.

We first recall the following definition from elementary number theory.

DEFINITION. A number n is said to be a *primitive root* of a prime q if the order of n modulo q is equal to $q - 1$.

5.25 LEMMA. *Let* p *and* q *be primes such that* p *is a primitive root of* q *, let* $H = H(p, q)$ *, and let* X *be a cyclic group of order* p *. If* r *is any positive integer dividing* $q - 1$ *, then there is an automorphism* ϕ *of* H *of order* r *which acts non-trivially on* H/H' *and a homomorphism* σ *from* H' *onto* X *such that* $\phi\sigma = \sigma$ *.*

Proof. The group H consists of matrices over the field F_1 obtained from $F_0 = \mathrm{GF}(p)$ by adjoining a primitive q-th root of unity, say λ . This element λ is a root of the polynomial

$$p(x) = x^{q-1} + x^{q-2} + \ldots + x + 1$$

of degree $q - 1$ over F_0 . The degree of F_1 over F_0 is equal to the order of p modulo q and, since p is a primitive root of q , this is equal to $q - 1$. It follows that $p(x)$ is the minimal polynomial of λ over F_0 .

Now F_1 is a splitting field for $p(x)$ and consequently the Galois group Γ of F_1 over F_0 is also the Galois group associated with the polynomial $p(x)$. As $F_1 \cong \mathrm{GF}(p^{q-1})$, the group Γ is cyclic of order $q - 1$, and since r divides $q - 1$ there is an element $\tau \in \Gamma$ of order r.

Define a mapping ϕ by the rule

$$\begin{pmatrix} \lambda^i & 0 \\ \alpha & 1 \end{pmatrix}^{\phi} = \begin{pmatrix} \lambda^i\tau & 0 \\ \alpha\tau & 1 \end{pmatrix}$$

for $0 \le i < q$ and $\alpha \in F_1$. As τ is a field automorphism and permutes the roots $\lambda, \lambda^2, \ldots, \lambda^{q-1}$ of $p(x)$, the mapping ϕ is an automorphism of H , and its order is obviously r .

The derived group of H consists of all matrices over F_1

having the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \quad (\alpha \in F_1) .$$

The mapping associating such a matrix with its bottom left-hand entry α is easily seen to be an isomorphism of H' onto the additive group of F_1. Now the elements $\lambda, \lambda^2, \ldots, \lambda^{q-1}$ form a basis for F_1 as a vector space over F_0. Hence if we write

$$a_i = \begin{pmatrix} 1 & 0 \\ \lambda^i & 1 \end{pmatrix}$$

for $1 \leq i \leq q-1$, then H' can be expressed as a direct product

$$H' = \langle a_1 \rangle \times \ldots \times \langle a_{q-1} \rangle .$$

We can therefore define a homomorphism σ from H' onto X by specifying that

$$a_i^\sigma = x$$

for $1 \leq i \leq q-1$, where x is a generator of X. Because τ permutes the elements $\lambda, \lambda^2, \ldots, \lambda^{q-1}$, the automorphism ϕ permutes the elements $a_1, a_2, \ldots, a_{q-1}$. From this we deduce that $\phi\sigma = \sigma$ as required. $\square$

In the next theorem we make use of the criterion provided by Lemma 2.43 (Chapter 2) for the existence of an automorphism of a treble product extending prescribed automorphisms on the three factors.

5.26 THEOREM. *Assume that the hypotheses of Theorem* 5.21 *are fulfilled with* $n = 1$ *and assume further that* p_1 *is a primitive root of* q_1. *If* p_2 *is any prime dividing* $q_1 - 1$ *then the group* G *of Theorem* 5.21 *may be constructed so as to have an automorphism* *of order* p_2 *which acts non-trivially on* G/G' .

Proof. Let $H = H(p_1, q_1)$ and define the group B and the homomorphism $\tau : C \to \operatorname{Aut} B$ as in the proof of Theorem 5.21.

Since $BC \cong H$ we can make use of Lemma 5.25 to construct the second homomorphism σ , as follows. By assumption A has an automorphism α of order p_1 : applying Lemma 5.25 with $X = \langle\alpha\rangle$ we deduce that there is a homomorphism

$$\sigma : B \to \operatorname{Aut} A$$

such that $B^{\sigma} = \langle\alpha\rangle$ and an automorphism ϕ of BC of order p_2 such that $\phi\sigma = \sigma$.

Let $G = \operatorname{Tr}(A, B, C; \sigma, \tau)$. By its construction, the group G has the properties of the treble product of Theorem 5.21. We now make use of Lemma 2.43 to produce an automorphism of G of order p_2 .

As ϕ is an automorphism of the split extension BC leaving both B and C invariant, we see from the Corollary to Lemma 2.43 that the restrictions β, γ of ϕ to B and C respectively satisfy

$$c^{\gamma\tau} = \beta^{-1}c^{\tau}\beta$$

for all $c \in C$. Also since $\phi\sigma = \sigma$ we have

$$b^{\beta\sigma} = b^{\phi\sigma} = b^{\sigma}$$

for all $b \in B$. Therefore the conditions of Lemma 2.43 are satisfied when we take the automorphism of A to be the identity automorphism.

We deduce from Lemma 2.43 that G has an automorphism ϕ_2 extending both β and γ and acting identically on A . Since both β and γ are automorphisms of order p_2 so is ϕ_2 . Also

$G/G' \cong C$, and γ is a non-trivial automorphism of C, so ϕ_2 acts non-trivially on G/G'. $\square$

The following rather cumbersome theorem is the main step in the construction of the rest of our examples.

5.27 THEOREM. *Let k be a positive integer and n either a positive integer or the symbol ∞, and let $q_0, q_1, \ldots, q_k$ and $p_1, p_2, \ldots, p_k$ be two finite sequences of primes such that*

(i) $p_i | q_{i-1}-1$ for $1 \le i \le k$,

(ii) p_i is a primitive root of q_i for $1 \le i \le k-1$,

(iii) $p_k \ne q_k$.

There is a soluble group $G = G_{k,n}$ with the following properties:

(a) the normal subgroups of G form a well-ordered chain consisting of the terms of the derived series and subgroups H with $G' \le H \le G$,

(b) G has derived length $2k+1$,

(c) $C_G\left(G^{(i)}/G^{(i+1)}\right) = G^{(i)}$, for $0 \le i \le 2k$,

(d) $G^{(2i)}/G^{(2i+1)}$ is an elementary abelian q_{k-i}-group, for $1 \le i \le k$, $G^{(2i+1)}/G^{(2i+2)}$ is an elementary abelian p_{k-i}-group, for $0 \le i \le k-1$, and G/G' is an abelian group of type $C_{q_k^n}$.

In the case $n = 1$, if p_k is a primitive root of q_k and if p_{k+1}, q_{k+1} are distinct primes such that $p_{k+1} | q_k-1$, then the group $G_{k,1}$ can be embedded in the group $G_{k+1,1}$ corresponding to the sequences of primes $q_0, \ldots, q_{k+1}$ and $p_1, \ldots, p_{k+1}$, such that

$$G''_{k+1,1} = G_{k,1}^{G_{k+1,1}} .$$

Proof. We prove by induction on k that the groups $G_{k,n}$ having properties *(a)*-*(d)* exist for every n and k, and that, if p_k is a primitive root of q_k and r is any positive integer dividing $q_k - 1$, then $G_{k,1}$ has an automorphism of order r which acts non-trivially on $G_{k,1}/G'_{k,1}$.

For the groups $G_{1,n}$ we may take the treble products $\mathrm{Tr}(A, B, C; \sigma, \tau)$ constructed as in Theorem 5.18, choosing A to be a cyclic group of order q_0, C to be a group of type $C_{q_1^n}$ (for the appropriate n) and B to be an elementary abelian p_1-group. These groups have properties *(a)*-*(c)* by Theorem 5.18, and also property *(d)* in view of the fact that

$$G'' = A^C = \underset{c \in C}{\mathrm{Dr}}\ A^c ,$$

$$G'/G'' \cong B ,$$

and

$$G/G' \cong C .$$

If further p_1 is a primitive root of q_1 and r is any positive integer dividing $q_1 - 1$ then by Theorem 5.26 we can arrange that $G_{1,1}$ has an automorphism of order r which acts non-trivially on $G_{1,1}/G'_{1,1}$.

Suppose now that $k > 1$. By induction there is a group $G_{k-1,1}$ with properties *(a)*-*(d)* associated with the sequences of primes $p_1, \ldots, p_{k-1}$ and $q_0, \ldots, q_{k-1}$. For simplicity of notation let us write $A = G_{k-1,1}$. As p_{k-1} is a primitive root of q_{k-1} and $p_k | q_{k-1}-1$, we may suppose further that A has an automorphism of order p_k which acts non-trivially on A/A' . Since A has property *(d)*, the factor A/A' has order q_{k-1} ; also properties *(a)* and *(c)* imply that A/A' is the only non-trivial central factor of A .

For each n with $1 \le n \le \infty$, let C_n be a group of type $C_{q_k^n}$ and let

$$W_n = A \text{ wr } C_n .$$

The conditions of Theorems 5.21 and 5.24 are satisfied with C replaced by C_n and the primes q_0, q_1, p_1 occurring in the statements of the theorems replaced by q_{k-1}, q_k, p_k respectively. We deduce that W_n can be embedded in a treble product

$$G_{k,n} = \mathrm{Tr}(A, B_n, C_n) ,$$

in which B_n is an elementary abelian p_k-group, such that the conclusions of Theorems 5.21 and 5.24 are simultaneously valid.

Now choose any n with $1 \le n \le \infty$, and write $G = G_{k,n}$. We show that G has properties *(a)*-*(d)*.

By Theorem 5.24 the only normal subgroups of G are the subgroups containing G' and subgroups of the form N^G , where $N \triangleleft A$. Because A has property *(a)*, the latter subgroups all have the form

$$(A^{(i)})^G$$

where $0 \le i \le 2k-1$. However, by Theorem 5.21 we have

$$G'' = \operatorname*{Dr}_{c \in C} A^c = A^G \tag{1}$$

so that

$$\begin{aligned}(A^{(i)})^G &= \operatorname*{Dr}_{c\in C} (A^{(i)})^c \\ &= (A^G)^{(i)} \\ &= G^{(i+2)} .\end{aligned}$$

Therefore every normal subgroup of G contained in G' coincides with some term of the derived series of G. Also, since $G/G' \cong C_n$, the subgroups containing G' form a well-ordered chain, and it follows that G has property *(a)*. Moreover equation (1) shows that the derived length of G exceeds that of A by 2, and hence is equal to $2k+1$. Thus G has property *(b)*.

To prove that G has property *(c)*, assume on the contrary that

$$C_G(G^{(i)}/G^{(i+1)}) > G^{(i)} \tag{2}$$

for some i with $0 \le i \le 2k$. We distinguish two cases.

Suppose first that $i \ge 3$. As G has property *(a)*, the inequality (2) implies that

$$G^{(i-1)} \le C_G(G^{(i)}/G^{(i+1)}) . \tag{3}$$

From (1) we see that each of the subgroups $G^{(i-1)}$, $G^{(i)}$, $G^{(i+1)}$ is expressible as a direct product

$$G^{(j)} = \operatorname*{Dr}_{c\in C} (A^{(j-2)})^c , \quad (i-1 \le j \le i+1) .$$

Hence (3) implies that

$$[A^{(i-3)}, A^{(i-2)}] \le A^{(i-1)}$$

which is a contradiction, for A has property *(c)*.

Now suppose that $i < 3$. As we remarked in the proof of Theorem 5.21 the factor-group $G/G''' = G/(A')^G$ is isomorphic to the treble product

$$T = \mathrm{Tr}(A/A', B, C)$$

formed using the natural action of B on A/A'. Since A/A' has order q_{k-1} and B acts non-trivially on it, T is a treble product of the type occurring in Theorem 5.18. Therefore, by Theorem 5.18, we have

$$C_T\left(T^{(i)}/T^{(i+1)}\right) = T^{(i)} .$$

However under the isomorphism between G/G''' and T the factor $G^{(i)}/G^{(i+1)}$ corresponds to $T^{(i)}/T^{(i+1)}$, so we deduce that

$$C_G\left(G^{(i)}/G^{(i+1)}\right) = G^{(i)}$$

which contradicts (2).

These contradictions show that G has property *(c)*.

To prove that G has property *(d)* we observe that

$$G/G' \cong C_n \text{, a group of type } C_{q_k}^{n} ,$$

$$G'/G'' \cong B_n \text{, an elementary abelian } p_k\text{-group,}$$

and, for $i > 1$,

$$G^{(i)}/G^{(i+1)} \cong \operatorname*{Dr}_{c \in C} \left(A^{(i-2)}/A^{(i-1)}\right)^c$$

and since A has property *(d)*, this implies that *(d)* also holds for the group G.

To complete the proof of the inductive step, suppose that $n = 1$ and that p_k is a primitive root of q_k. Then by Theorem 5.26 we can construct $G_{k,1} = \mathrm{Tr}\left(A, B_1, C_1\right)$ so that for each positive integer r dividing $q_k - 1$ there is an automorphism of $G_{k,1}$ of order r which acts non-trivially on $G_{k,1}/G'_{k,1}$.

Hence by induction the groups $G_{k,n}$ can be constructed and have properties *(a)*-*(d)* for every k and n. The final assertion of the theorem is now a consequence of the manner of construction of the groups $G_{k,n}$. □

5.28 Remark. It is easy to see that sequences of primes satisfying conditions *(i)* and *(ii)* of Theorem 5.27 do exist. For example, take $p_1 = p_2 = \ldots = 2$, choose q_0 to be any odd prime, and for $n \geq 1$ let q_n be a prime such that 2 is a primitive root of q_n (e.g. 3, 5, 11, 13, 29, ...).

Unfortunately it is still, as far as we know, an unsolved problem whether 2 (or any other prime) is a primitive root of infinitely many primes. For conjectures of Artin concerning this, and some evidence to support them, we refer to Shanks [36].

For our purposes, however, it is enough to know that there are at least two choices for q_n for each value of n in the above. This makes it possible to choose the primes so that

$$q_0, p_1, q_1, p_2, q_2, \ldots$$

is not a periodic sequence. (We can also avoid the prime 2 by taking $p_n = 3$ and $q_n = 7$ or 19 for every n .) We shall make use of this fact later.

We can now establish the existence of quasi-radicable soluble groups satisfying Min-n of arbitrarily large derived (and nilpotent) lengths.

5.29 THEOREM. *For every positive integer d there is a quasi-radicable soluble group of derived length d whose normal subgroups form a chain of order type $\omega + 1$, consisting of the terms of the derived series and the subgroups containing the derived group. Moreover the derived series of each of these groups is also the shortest series with nilpotent factors.*

Proof. Since all the above properties are inherited by non-trivial homomorphic images, it is enough to exhibit a soluble group with these properties having derived length d, for every odd positive integer $d \geq 3$.

Suppose $d = 2k + 1$, where $k \geq 1$, and let $q_0, \ldots, q_k$ and $p_1, \ldots, p_k$ be any two sequences of primes satisfying the conditions *(i)* and *(ii)* of Theorem 5.27. Let $G = G_{k,\infty}$, the group associated with these sequences by Theorem 5.27 in the case $n = \infty$. Then G has properties *(a)-(d)*.

By property *(b)*, G has derived length d. Also property *(a)* shows that the only normal subgroups of G are the terms of the derived series and the subgroups containing G'. As G/G' is a quasicyclic group (by property *(d)*), the latter subgroups form a chain of order type $\omega + 1$, and so therefore do the normal subgroups of G. The fact that the derived series is the shortest series with nilpotent factors follows from *(c)*. □

5.3 Locally Soluble Groups satisfying Min-n

We next apply Theorem 5.27 to construct perfect locally soluble groups satisfying Min-n with similar properties to the group constructed by McLain [26].

5.31 THEOREM. *Let $q_0, q_1, q_2, \ldots$ and $p_1, p_2, \ldots$ be two infinite sequences of (not necessarily distinct) primes such that*

(i) $p_i | q_{i-1}-1$, *for* $i = 1, 2, \ldots$,

(ii) p_i *is a primitive root of* q_i, *for* $i = 1, 2, \ldots$.

There is a group G with the following properties:

(a) G is a periodic locally soluble group,

(b) *the normal subgroups of* G *form a well-ordered chain*

$$1 = H_0 < H_1 < H_2 < \dots < H_\omega = G$$

of order type $\omega + 1$,

(c) G *is perfect,*

(d) H_{2i+1}/H_{2i} *is an elementary abelian* q_i*-group, for* $i = 0, 1, 2, \dots$, *and* H_{2i}/H_{2i-1} *is an elementary abelian* p_i*-group, for* $i = 1, 2, \dots$.

Before we prove this theorem it is convenient to state the following extremely simple lemma, with a corollary. Since the lemma is an almost immediate consequence of the definition of the treble product, we omit the proof.

5.32 LEMMA. *Let* G *be a group having subgroups*

$$G_0 \le G_1 \le G_2 \le \dots$$

such that $G = \bigcup_{i \ge 0} G_i$, *and such that for each* $i \ge 1$ *the subgroup* G_i *is expressible as a treble product*

$$G_i = \mathrm{Tr}(G_{i-1}, B_i, C_i)$$

for certain subgroups B_i, C_i *of* G .

For each $i \ge 0$, *let* X_i *be the set of elements of* G *expressible in the form*

$$c_{i+1}\, c_{i+2} \dots c_{i+r} ,$$

where $c_j \in C_j$ *for each* j , *and* $r \ge 1$.

If $N \triangleleft G_i$ *for some* $i \ge 0$, *and if* $N^{B_j} \le N$ *whenever* $j \ge i$, *then*

$$N^G = \mathop{\mathrm{Dr}}_{x \in X_i} N^x . \quad \square$$

5.33 COROLLARY. *With the hypotheses of the lemma, we have*

$$N^G \cap G_i = N .$$

Proof of Corollary. Apply the lemma with N replaced by G_i, and we obtain

$$\begin{aligned} G_i^G &= \operatorname*{Dr}_{x\in X_i} G_i^x \\ &= G_i \times G_i^* \text{, say,} \end{aligned}$$

where $G_i^* = \operatorname*{Dr}_{1\neq x\in X_i} G_i^x$.

Hence we have

$$\begin{aligned} G_i \cap N^G &= G_i \cap \operatorname*{Dr}_{x\in X_i} N^x \\ &\le G_i \cap (N\times G_i^*) \\ &= N(G_i \cap G_i^*) \text{, by the modular law,} \\ &= N . \end{aligned}$$

Since the reverse inclusion is obvious, we have the result. □

Proof of Theorem 5.31. For each positive integer k, let $G_k = G_{k,1}$ be the group associated with the sequences of primes $q_0, \ldots, q_k$ and $p_1, \ldots, p_k$ by the construction of Theorem 5.27. Since p_i is a primitive root of q_i for every $i \ge 1$, these groups can be constructed so that

$$G_1 \le G_2 \le G_3 \le \ldots \tag{1}$$

and

$$G''_{i+1} = G_i^{G_{i+1}} \tag{2}$$

for all $i \ge 1$.

Let $G = \bigcup_{i\ge 1} G_i$ be the direct limit of the ascending sequence of groups (1). Then G is a union of finite soluble groups and is therefore periodic and locally soluble. Also (2) clearly implies that G is perfect.

Define normal subgroups $H_0, H_1, H_2, \ldots$ of G by

$$H_{2i+1} = G_i^G ,$$

$$H_{2i} = (G_i')^G ,$$

for $i = 0, 1, 2, \ldots$. By Lemma 5.32 there are subsets

$$X_1 \subset X_2 \subset X_3 \subset \ldots \subseteq G$$

such that

$$H_{2i+1} = \operatorname*{Dr}_{x \in X_i} G_i^x ,$$

$$H_{2i} = \operatorname*{Dr}_{x \in X_i} (G_i')^x .$$

The subgroups H_i clearly form an ascending chain

$$1 = H_0 < H_1 < H_2 < \ldots$$

such that

$$H_\omega = \bigcup_{i \geq 1} H_i = G .$$

Now

$$H_{2i+1}/H_{2i} \cong \operatorname*{Dr}_{x \in X_i} G_i^x/(G_i')^x ,$$

which is a direct product of groups isomorphic to G_i/G_i' . By Theorem 5.27, G_i/G_i' is a cyclic group of order q_i . Hence H_{2i+1}/H_{2i} is an elementary abelian q_i-group, for $i = 0, 1, 2, \ldots$.

In view of equation (2) we have also

$$H_{2i-1} = G_{i-1}^G = (G_i'')^G$$

so that

$$H_{2i}/H_{2i-1} \cong \operatorname*{Dr}_{x \in X_i} (G_i')^x/(G_i'')^x ,$$

which is a direct product of groups isomorphic to G_i'/G_i'' . Hence, by Theorem 5.27, H_{2i}/H_{2i-1} is an elementary abelian p_i-group, for each $i \geq 1$.

It remains to show that the subgroups H_i are the only normal subgroups of G . For each $i \geq 0$, Theorem 5.27 shows that the only normal subgroups of G_i are the subgroups $G_i^{(j)}$, where $0 \leq j \leq 2i+1$. By (2) these are precisely the subgroups

$$G_k^{G_i} , \quad (G_k')^{G_i}$$

for $0 \leq k \leq i$, and by Corollary 5.33 we have

$$G_k^{G_i} = G_k^G \cap G_i = H_{2k+1} \cap G_i$$

and

$$(G_k')^{G_i} = (G_k')^G \cap G_i = H_{2k} \cap G_i .$$

Thus the only normal subgroups of G_i are the subgroups $H_j \cap G_i$, for $j \leq 2i+1$.

Suppose now that N is any proper normal subgroup of G . Then there is an integer $j \geq 0$ such that

$$H_j \leq N \tag{3}$$

but

$$H_{j+1} \nleq N . \tag{4}$$

Suppose, if possible, that $H_j < N$. Then for some integer $k \geq j$ we have

$$H_j \cap G_i < N \cap G_i \tag{5}$$

for all $i \geq k$. But $N \cap G_i \triangleleft G_i$, so there is an integer $r(i)$ such that $N \cap G_i = H_{r(i)} \cap G_i$, and (5) implies that $r(i) \geq j+1$, so we have

$$H_{j+1} \cap G_i \leq N \cap G_i$$

for all $i \geq k$. Hence

$$\begin{aligned} H_{j+1} &= \bigcup_{i \geq k} H_{j+1} \cap G_i \\ &\leq \bigcup_{i \geq k} N \cap G_i \\ &= N , \end{aligned}$$

contradicting (4). Therefore $N = H_j$ and, since N was an arbitrary proper normal subgroup of G , this completes the proof. □

The group constructed by McLain also has properties *(a)-(c)* of Theorem 5.31. This group is a $\{2, 3\}$-group and has, as we have already mentioned, only two isomorphism classes of non-trivial homomorphic images. The next result shows that Theorem 5.31 enables one to construct a large number of groups with properties *(a)-(c)* which are not isomorphic to McLain's group.

5.34 THEOREM. *There are* $2^{\aleph_0}$ *pairwise non-isomorphic countable groups with properties (a)-(c) of Theorem* 5.31. *Among these groups there are* $2^{\aleph_0}$ *Hopfian groups.*

Proof. Obviously two groups constructed using the procedure involved in the proof of Theorem 5.31 cannot be isomorphic unless they are both associated with the same two sequences of primes. Moreover any group constructed in this way is isomorphic to a proper factor-group if and only if the associated prime sequences $q_0, q_1, \ldots$ and $p_1, p_2, \ldots$ are such that the alternating sequence

$$q_0, p_1, q_1, p_2, q_2, \ldots \tag{1}$$

is periodic.

Hence to show that there are (at least) $2^{\aleph_0}$ pairwise non-isomorphic groups with properties *(a)-(c)* of Theorem 5.31 it is sufficient to prove that there are $2^{\aleph_0}$ distinct pairs of sequences of primes satisfying the conditions *(i)* and *(ii)* of Theorem 5.31. Moreover since only countably many such pairs of sequences can be such that the alternating sequence (1) is periodic, this would prove in addition the existence of $2^{\aleph_0}$ pairwise non-isomorphic Hopfian groups with properties *(a)-(c)*.

However we have already indicated in Remark 5.28 that by taking $p_i = 2$ and $q_i = 3$ or 5 for each i we obtain sequences satisfying *(i)* and *(ii)*; and there are clearly $2^{\aleph_0}$ possible choices using these values.

The groups obtained in this way are all countable, and since there are only $2^{\aleph_0}$ isomorphism classes of countable groups, the total number of countable groups with properties *(a)-(c)* is precisely $2^{\aleph_0}$.

5.4 A Note on Treble Product Towers

For the construction of the examples in Sections 5.2 and 5.3 we made use of a procedure for forming iterated treble products, relying on Lemma 2.43 to define automorphisms of the groups constructed at the successive stages of the iteration. A different iterative procedure is used by Heineken and Wilson in [13] for constructing *treble product towers*, which they use to produce examples of torsion-free locally soluble groups satisfying Min-n. In this section we give a brief description of the treble product tower, and compare the iterative procedure used for its construction with that we have used for constructing our examples.

The data for the construction of the treble product tower consists of a family of groups

$$\{A_\sigma : 0 \le \sigma < \rho\} ,$$

indexed by some ordinal ρ , and a homomorphism

$$\theta_{\sigma+1} : A_{\sigma+1} \to \operatorname{Aut} A_\sigma$$

for each ordinal σ with $1 \le \sigma+1 < \rho$. To simplify our description, however, we assume that $\rho = \omega$, since this is the case of most interest to us. In this case the treble product tower

$$\operatorname{Trt}(A_n : 0 \le n < \omega)$$

of the groups $A_0, A_1, A_2, \ldots$ may be constructed from the above data as follows.

We first define two sequences of groups $K_1, K_2, \ldots$ and $L_0, L_1, L_2, \ldots$ and a sequence of homomorphisms

$$\phi_i : A_i \to \operatorname{Aut} L_i \quad (i = 0, 1, 2, \ldots)$$

such that, for each $i > 0$, K_i is a split extension

$$K_i = L_{i-1} \underset{\phi_{i-1}}{]} A_{i-1} .$$

We start by taking $K_1 = A_0$, $L_0 = 1$ and defining ϕ_0 to be the unique homomorphism $A_0 \to \operatorname{Aut} L_0$. Assuming inductively that the groups K_i, L_{i-1} and the homomorphism ϕ_{i-1} have been constructed for $1 \le i \le n$, we define

$$K_{n+1} = \operatorname{Tr}(L_{n-1}, A_{n-1}, A_n; \phi_{n-1}, \theta_n) .$$

Then K_n is a subgroup of K_{n+1} , and K_{n+1} is a split extension of the subgroup

$$K_n^{K_{n+1}} = L_{n-1}^{A_n} A_{n-1}$$

by the group A_n . We set

$$L_n = K_n^{K_{n+1}}$$

and define $\phi_n : A_n \to \mathrm{Aut}\, L_n$ to be the homomorphism for which we have $K_{n+1} = L_n \,]_{\phi_n}\, A_n$.

In this way the groups K_i are defined for each positive integer i and form an ascending chain

$$K_1 \le K_2 \le K_3 \le \ldots .$$

The treble product tower $\mathrm{Trt}\left(A_n : 0 \le n < \omega\right)$ of the groups $A_0, A_1, A_2, \ldots$ is defined to be the direct limit

$$\bigcup_{n \ge 1} K_n$$

of the groups K_i .

From this description it is clear that the iterative procedure used to define the treble product tower is different from that we used to construct our examples.

In applying the treble product tower construction to construct torsion-free locally soluble groups satisfying Min-n , Heineken and Wilson take each of the groups A_i to be a torsion-free divisible abelian group and define homomorphisms

$$\theta_{i+1} : A_{i+1} \to \mathrm{Aut}\, A_i$$

so that, for each i , A_i becomes a faithful irreducible A_{i+1}-module. Moreover this is done in such a way that, for each i , the treble product

$$T_i = \mathrm{Tr}(A_i, A_{i+1}, A_{i+2}; \theta_{i+1}, \theta_{i+2})$$

has the property that the normal closure $A_i^{T_i}$ is a minimal normal subgroup of T_i, for each i. From this they deduce that every proper normal subgroup of the treble product tower $G = \mathrm{Trt}(A_n : 0 \leq n < \omega)$ has the form K_i^G, where the groups K_i are defined as above.

Unfortunately we cannot, on the basis of the results we have proved, use the treble product tower in this way for the construction of periodic locally soluble groups of the type considered in section 5.3. The reason for this is that Lemma 5.11 is applicable only to *cyclic* minimal normal subgroups (see the example immediately following Lemma 5.11). Hence in order to invoke this lemma to prove that each of the treble products T_i defined above has the subgroup $A_i^{T_i}$ as a minimal normal subgroup, we should need to take each of the groups A_i to be a cyclic group. If in addition A_i is to be a faithful irreducible A_{i+1}-module for each i, then A_i must have prime order, p_i say, for each i, and the primes p_i must satisfy $p_{i+1} | p_i - 1$, for each i, since $|\mathrm{Aut}\, A_i| = p_i - 1$. However this implies that

$$p_1 > p_2 > p_3 > \ldots ,$$

which is clearly impossible.

REFERENCES

[1] Reinhold Baer, "Nilgruppen", *Math. Z.* 62 (1955), 402-437.

[2] Reinhold Baer, "Irreducible groups of automorphisms of abelian groups", *Pacific J. Math.* 14 (1964), 385-406.

[3] V.S. Čarin, "A remark on the minimal condition for subgroups" (Russian), *Dokl. Akad. Nauk. SSSR (N.S.)* 66 (1949), 575-576.

[4] S.N. Černikov, "Infinite locally soluble groups" (Russian), *Mat. Sb.* 7 (1940), 35-64.

[5] S.N. Černikov, "Complete groups with ascending central series" (Russian), *Mat. Sb.* 18 (60) (1946), 397-422.

[6] L. Fuchs, *Infinite abelian groups*, Vol. I, (Pure and Applied Mathematics, v. 36, Academic Press, N.Y., 1970).

[7] A.D. Gardiner, B. Hartley and M.J. Tomkinson, "Saturated formations and Sylow structure in locally finite groups", *J. Algebra* 17 (1971), 177-211.

[8] V.M. Gluškov, "On the theory of *ZA*-groups" (Russian), *Dokl. Akad. Nauk. SSSR (N.S.)* 74 (1950), 885-888.

[9] P. Hall, "Finiteness conditions for soluble groups", *Proc. London Math. Soc.* (3) 4 (1954), 419-436.

[10] P. Hall, "On the finiteness of certain soluble groups", *Proc. London Math. Soc.* (3) 9 (1959), 595-622.

[11] P. Hall, "The Frattini subgroups of finitely generated groups", *Proc. London Math. Soc.* (3) 11 (1961), 327-352.

[12] B. Hartley and D. McDougall, "Injective modules and soluble groups satisfying the minimal condition for normal subgroups", *Bull. Austral. Math. Soc.* 4 (1971), 113-135.

[13] H. Heineken and J.S. Wilson, "Locally soluble groups with Min-n ", *J. Austral. Math. Soc.* (to appear).

[14] K.A. Hirsch, "On infinite soluble groups (I)", *Proc. London Math. Soc.* (2) **44** (1938), 53-60.

[15] K.A. Hirsch, "On infinite soluble groups (II)", *Proc. London Math. Soc.* (2) **44** (1938), 336-344.

[16] K.A. Hirsch, "On infinite soluble groups (III)", *Proc. London Math. Soc.* (2) **49** (1946), 184-194.

[17] K.A. Hirsch, "On infinite soluble groups (IV)", *J. London Math. Soc.* **27** (1952), 81-85.

[18] K.A. Hirsch, "On infinite soluble groups (V)", *J. London Math. Soc.* **29** (1954), 250-251.

[19] B. Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, Band 134; Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[20] A.G. Kurosh, *The theory of groups*, Vol. II, 2nd edition, (translated from the Russian by K.A. Hirsch; Chelsea, New York, 1960).

[21] A.G. Kuroš and S.N. Černikov, "Solvable and nilpotent groups" (Russian), *Uspehi Mat. Nauk.* **2** no. **3** (**19**) (1947), 18-57; *Amer. Math. Soc. Transl.* **80** (1953).

[22] William Judson LeVeque, *Topics in number theory*, Vol. I (Addison-Wesley, Reading, Massachusetts; 1956).

[23] A.I. Mal'cev, "On certain classes of infinite soluble groups" (Russian), *Mat. Sb. (N.S.)* **28** (**70**) (1951), 567-588; *Amer. Math. Soc. Transl.* (2) **2** (1956), 1-21.

[24] David McDougall, "Soluble groups with the minimum condition for normal subgroups", *Math. Z.* **118** (1970), 157-167.

[25] D.H. McLain, "On locally nilpotent groups", *Proc. Cambridge Philos. Soc.* **52** (1956), 5-11.

[26] D.H. McLain, "Finiteness conditions in locally soluble groups", *J. London Math. Soc.* **34** (1959), 101-107.

[27] B.H. Neumann, "Ascending derived series", *Compositio Math.* **13** (1956), 47-64.

[28] B.H. Neumann, "Twisted wreath products of groups", *Arch. Math.* **14** (1963), 1-6.

[29] B.H. Neumann and Hanna Neumann, "Embedding theorems for groups", *J. London Math. Soc.* **34** (1959), 465-479.

[30] Peter M. Neumann, "On the structure of standard wreath products of groups", *Math. Z.* **84** (1964), 343-373.

[31] M.F. Newman, "On a class of metabelian groups", *Proc. London Math. Soc.* (3) **10** (1960), 354-364.

[32] Ya. D. Polovickiĭ, "Layerwise extremal groups" (Russian), *Dokl. Akad. Nauk. SSSR* **134** (1960), 533-535; *Soviet Math. Dokl.* **1** (1960), 1112-1113.

[33] Derek J.S. Robinson, *Infinite soluble and nilpotent groups* (Queen Mary College Mathematics Notes, London, ND, [1968]).

[34] Derek J.S. Robinson, *Finiteness conditions and generalized soluble groups,* Part 1 (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62; Springer-Verlag, Berlin, Heidelberg, New York, 1972).

[35] Eugene Schenkman, *Group theory* (Van Nostrand, Princeton, New Jersey; Toronto; New York; London; 1965).

[36] Daniel Shanks, *Solved and unsolved problems in number theory,* Vol. 1 (Spartan Books, Washington D.C., 1962).

[37] John S. Wilson, "Some properties of groups inherited by normal subgroups of finite index", *Math. Z.* **114** (1970), 19-21.